Block-Spin Distributions for Short-Range Attractive Ising Models.

G. Gallavotti

Instituut voor Theoretische Fysica, Universiteit Nijmegen - Nijmegen

A. Martin-Löf

Mathematics Department, University of Uppsala - Uppsala

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**Summary.** — We give some rigorous results on the block-spin distributions for \((h, \beta) \neq (0, \beta_c)\). Under some assumptions on the critical correlation functions we speculate about the block-spin distributions at the critical point.

1. Introduction.

Let \(Z^d\) be the \(d\)-dimensional square lattice and divide it into disjoint squares with side \(L\).

If \(\xi = (x_1, \ldots, x_d) \in Z^d\), the square \(n_i L < x_i < (n_i + 1) L\), \(i = 1, \ldots, d\), will be denoted by \((n, L)\); here \(n_i\) are integers and \(n\) denotes the \(d\)-ple \((n_1, \ldots, n_d)\).

We are interested in the random variables \(\{\text{block spins}\}\)

\[
\nu_n = \frac{\sum_{\xi \in (n, L)} \sigma_\xi - \left< \sum_{\xi \in (n, L)} \sigma_\xi \right>}{L^{d+1}},
\]

where \(\sigma_\xi = \pm 1\) is a Gibbs random field \((1)\) associated with an equilibrium state.

of the formal Hamiltonian

\[(1.2) \quad H(\sigma) = -\beta \sum_{(x,y)} J(x-y) \sigma_x \sigma_y - \beta h \sum_{x} \sigma_x, \]

and the average (1.1) is supposed to be taken with respect to the probability distribution of the just mentioned random field; \( \varrho \) is a parameter such that \( 0 < \varrho < 2 \).

In (2) it has been shown that for \( \varrho = 1 \) and \( h \) large the joint distribution function of a finite set of block spins \( \{v_n\}_{n \in A} \) has a simple asymptotic form in the limit \( L \to \infty \). More precisely, it is the characteristic function

\[(1.3) \quad \langle \exp \left[ i \sum_{n \in A} \omega_n v_n \right] \rangle, \]

which has a simple asymptotic expansion as \( L \to \infty \).

In this connection we remark that the existence of a limit for (1.3) when \( L \to \infty \) (and \( h \) is large) is a quite trivial consequence of the central-limit theorem for Gibbs processes (3); the interesting part of the statement is about the corrections for \( L \) large and about their interpretation (2).

The first purpose of this paper is to show that asymptotic expansion for (1.3) given in (2) extends to the whole region of the \((\beta, h)\)-plane with \( h \neq 0 \) provided \( J(x) > 0 \) and \( J(x) = 0 \) for \( |x| > \lambda \), which are conditions we hereafter assume valid.

The case \( h = 0 \) is more difficult and we have no new results about this case. It is however known that under the assumption of exponential decay of \( \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle \) as \(|x-y| \to \infty \) the average central-limit theorem holds so that the limit of (1.3) exists (3), but we are unable to estimate the corrections for \( L \) large in a meaningful way.

The second purpose of this paper is to try to investigate the block spins in the case \( \beta = \beta_c, \ h = 0 \) (here \( \beta_c \) = critical temperature).

Because of obvious physical reasons the choice of the variable \( \varrho \) should be made, if possible, according to the criterion (2) that

\[(1.4) \quad 0 < \lim_{L \to \infty} \langle v_0^2 \rangle < \infty. \]

As the reader may expect, we are only able to investigate the consequences of a number of assumptions on the decay rate of the Ursell functions for the random field at \((\beta_c, 0)\) and in its neighbourhood. This discussion leads to several

consequences. For instance we construct, as a by-product, a vast family of «fixed points» for a renormalization transformation associated with the block spins.

Since only few of the assumptions on the Ursell functions can be actually verified in nontrivial cases, we regard this second part of the paper mainly as a heuristic investigation of some of the pathologies that may arise at the critical point.

2. – Some rigorous results.

In this Section we summarize a few known definitions and basic results. Let $M \subset \mathbb{Z}^d$ be a finite square box (centred at the origin). If $h_{\xi}$ is a complex-valued function defined on $\mathbb{Z}^d$, we put

$$Z_M(h) = \sum_{\sigma_{\xi} \in \mathbb{M}} \exp \left[ \beta \sum_{\xi \in M} J(\xi - \xi') \sigma_{\xi} \sigma_{\xi'} + \beta \sum_{\xi} h_{\xi} \sigma_{\xi} \right];$$

as usual we shall also regard $Z_M(h)$ as a function of the variables $\zeta_x = \exp \left[ -\beta h_x \right]$. We remember that $J(\xi) > 0$ is assumed everywhere. Define

$$u_{\xi, \ldots, \xi_k}(\xi_1, \ldots, \xi_k; h) = \frac{\partial^k \log Z_M(h)}{\partial \beta h_{\xi_1} \cdots \partial \beta h_{\xi_k}}.$$

If $h_{\xi} = h$, $\xi \subset M$ and $h$ is real, we shall denote by $u_{\xi}(\xi_1, \ldots, \xi_k)$ the limit as $M \to \infty$ of $u_{\xi}(\xi_1, \ldots, \xi_k; h)$ (the $h$-dependence will be added as an index if necessary).

The functions

$$u_{\xi}(\xi_1, \ldots, \xi_k) = \lim_{M \to \infty} u^{(M)}_{\xi}(\xi_1, \ldots, \xi_k; h),$$

when $h_{\xi} = h$, $h$ real, will be referred to as the Ursell functions (the limits (2.3) are known to exist as a consequence of the GKS inequalities (4)).

Theorem 1. Let $D$ be the open unit circle of the complex plane. If $C \subset D$ is closed there exist constants $A_{\xi}(C)$, $\kappa_{\xi}(C)$ such that if $\zeta_{\xi} \in C$, $\forall \xi \in M$

$$|u^{(M)}_{\xi}(\xi_1, \ldots, \xi_k; h)| \leq A_{\xi}(C) \exp \left[ -\kappa_{\xi}(C) \Delta(\xi_1, \ldots, \xi_k) \right],$$

where $A(\xi_1, \ldots, \xi_k) =$ length of the shortest graph which connects all the points $\xi_1, \ldots, \xi_k$.

Proof. This result is due to Pfister. See Appendix.

Theorem 2. If $h_\xi > 0$, $\forall \xi \in M$, there is a constant $C_k$ such that

\begin{equation}
|u_2^{(M)}(\xi_1, \ldots, \xi_k|\mathbf{h})| \leq C_k \prod_{i=1}^k f(d_i)^{\nu_k},
\end{equation}

where $d_i =$ distance between $\xi_i$ and the set $(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_M)$ and

\begin{equation}
f(d) = \max_{|\xi - \xi'| \geq d} u_2^{(M)}(\xi, \xi'|\mathbf{h}).
\end{equation}

Furthermore

\begin{equation}
0 < u_2^{(M)}(\xi, \xi'|\mathbf{h}) < u_2^{(M)}(\xi, \xi'|0) < u_2(\xi, \xi').
\end{equation}

Proof. See (4.5).

Theorem 3. In the case of the 2-dimensional nearest-neighbour Ising model the function $u_2(\xi, \xi')$ is such that if $\beta = \beta_c$ and $h = 0$

\begin{equation}
u_2(\xi, \xi') = \frac{A}{|\xi - \xi'|^4} \left(1 + O(|\xi - \xi'|^{-1})\right),
\end{equation}

provided $\xi, \xi'$ are on the same lattice line (horizontal or vertical).

Proof. See (e).

Theorem 4. Define for $\beta = \beta_c$ and $h = 0$

\begin{equation}X_2(R) = \sum_{|\xi| \leq R} u_2(0, \xi),\end{equation}

and for $\beta = \beta_c$, $h = h$ the ($\xi$-independent) magnetization

\begin{equation}M(h) = u_1(\xi).
\end{equation}

Assume that the following limits exist:

\begin{equation}\frac{1}{\delta} = \lim_{h \to \delta^+} \frac{\log M(h)}{\log h},\end{equation}

\begin{equation}2 - \eta = \lim_{h \to 0} \frac{\log X_2(R)}{\log R}.
\end{equation}


Then

\[ 2 - \eta < d \frac{\delta - 1}{\delta + 1}. \]

**Proof.** See (4,7,8).

3. - The asymptotic distribution of the block spins for \( h \neq 0 \).

This Section contains the proof of our main result. Let \( \varphi \) be a bounded, piecewise smooth function on \( \mathbb{R}^d \), with compact support. Consider the variables \( v(\varphi), \vartheta(\varphi) \) implicitly defined by

\[ v(\varphi) = \vartheta(\varphi) - \langle \vartheta(\varphi) \rangle = L^{-d\alpha_1/2} \left( \sum_{\xi \in \mathbb{Z}^d} \varphi \left( \frac{\xi}{L} \right) \sigma_{\xi} - \sum_{\xi \in \mathbb{Z}^d} \varphi \left( \frac{\xi}{L} \right) \langle \sigma_{\xi} \rangle \right), \]

where \( L \) is a positive integer and, if \( \xi = (x_1, \ldots, x_d) \),

\[ \frac{\xi}{L} = \left( \frac{x_1}{L}, \ldots, \frac{x_d}{L} \right) \in \mathbb{R}^d. \]

The parameter \( q \) will be set equal to 1 throughout this Section. The average \( \langle \cdots \rangle \) is with respect to the random field of Sect. 1.

We shall seek an asymptotic expression (for \( L \to \infty \)) for the Fourier transform of the probability distribution of \( v(\varphi) \), i.e. for its characteristic function

\[ \langle \exp [itv(\varphi)] \rangle \]

under the assumption that the parameter \( h \) characterizing the random field is \( \neq 0 \) (the parameter \( \beta \) is arbitrary, hence, in particular, the results below hold for \( \beta = \beta_\varepsilon \)).

The average (3.2) can be computed as the limit as \( M \to \infty \) of

\[ \sum_{\sigma_{\xi_1}, \ldots, \sigma_{\xi_M}} \exp [itv(\varphi)] \exp \left[ \beta \sum_{\xi \in \mathbb{Z}^d} J(\xi - \xi') \sigma_{\xi} \sigma_{\xi'} + \beta h \sum_{\xi} \sigma_{\xi} \right] Z_M(h), \]

where \( h = h, \xi \neq M \). This is another consequence of the GKS inequalities and of the fact that \( \exp [itv(\varphi)] \) can be expressed as a finite linear combination of products of \( \sigma_{\xi} \)'s (because \( \sigma_{\xi}^2 = 1 \) (4). We shall denote the ratio in (3.3) as \( \langle \exp [itv(\varphi)] \rangle_{(M)} \).

Clearly (3.1) and (3.3) imply that, if we define 
\[ \delta h = i L^{d/2} q(\xi/L), \]
then
\begin{equation}
\langle \exp[i \hat{t} \hat{q}(q)] \rangle_{(M)} = \frac{Z_M(\hat{h} + t \delta \tilde{h})}{Z_M(\hat{h})}.
\end{equation}

Therefore, if \( h > 0 \),
\begin{equation}
\langle \exp[i \hat{t} \hat{q}(q)] \rangle_{(M)} = \exp \left[ \int_0^t \frac{\partial}{\partial s} \log Z_M(\hat{h} + s \delta \tilde{h}) \, ds \right];
\end{equation}

this formula is obvious if one remembers the Lee-Yang theorem (\(^9\)) and the fact that \( \delta \tilde{h} \) is purely imaginary, which implies that \( Z_M \) has no zeros while \( s \) goes from 0 to 1.

Next we develop the function under the integral sign in a Taylor series up to order \( k - 1 \) and express the remainder in the Lagrange form: using (2.2) we find as a result
\begin{equation}
\langle \exp[i \hat{t} \hat{q}(q)] \rangle_{(M)} + \sum_{p=1}^k \frac{(it)^p}{p!} L^{-p d/2} \sum_{\xi_1, \ldots, \xi_p} u^{(n)}(\xi_1, \ldots, \xi_p) \varphi \left( \frac{\xi_1}{L} \right) \ldots \varphi \left( \frac{\xi_p}{L} \right) + \ldots + \frac{(it)^{k+1}}{(k+1)!} L^{-d(k+1)/2} \sum_{\xi_1, \ldots, \xi_{k+1}} \{i^{k+1} u_{k+1}(\xi_1, \ldots, \xi_{k+1} | \hat{h} + s \delta \tilde{h}) \},
\end{equation}

where \( u^{(n)}(\ldots) \) shortens \( u^{(n)}(\xi_1, \ldots, \xi_p | \hat{h}) \) and
\begin{equation}
\{i^{k+1} u_{k+1}^{(n)}(\xi_1, \ldots, \xi_{k+1} | \hat{h} + s \delta \tilde{h}) \} = \text{Re} \, i^{k+1} u_{k+1}^{(n)}(\xi_1, \ldots, \xi_{k+1} | \hat{h} + s \delta \tilde{h}) + \text{Im} \, i^{k+1} u_{k+1}^{(n)}(\xi_1, \ldots, \xi_{k+1} | \hat{h} + s \delta \tilde{h})
\end{equation}

with \( 0 < s, \tau, s \tau < t \).

Using Theorem 1 with \( C = \text{circle with radius } \exp[- \beta |h|] < 1 \), we find that the last term in (3.6) can be bounded by
\begin{equation}
L^{-d(k+1)/2} B \rho M^k D_k^{k+1},
\end{equation}

where \( D_k \) is a suitable constant (\( M \) independent), \( B \rho \) is a measure of the support of \( \varphi \), \( M \rho = \text{maximum of } |\varphi| \).

The uniformity in \( M \) of the bound (3.7) allows us to draw the conclusion, in the limit \( M \to \infty \), that
\begin{equation}
\langle \exp[i \hat{t} \hat{q}(q)] \rangle = \exp \left[ \sum_{p=1}^k \frac{(it)^p}{p!} L^{-p d/2} \sum_{\xi_1, \ldots, \xi_p} u_p(\xi_1, \ldots, \xi_p) \prod_{j=1}^p \varphi \left( \frac{\xi_j}{L} \right) + \theta L^{-d(k+1)/2} B \rho M^k D_k^{k+1} \right],
\end{equation}

where \( \theta \) is an unknown complex function, \( |\theta| < 1 \).

This is our main result. Let us discuss its implications.

\(^9\) See, for instance, (\(^9\)).
1) Formula (3.8) implies the mean central-limit theorem for the variable \( v(\varphi) \): the distribution of \( v(\varphi) \) converges weakly to a Gaussian with dispersion

\[
\left( \sum \frac{u_x(0, \xi)}{\xi} \right) \int_{R^d} f(x)^2 \, dx.
\]

This is because in the limit \( L \to \infty \) only the term with \( q = 2 \) survives in (3.8).

2) Let \( A = (n_1, \ldots, n_N) \) be a finite subset of \( Z^d \) and let \( \omega_{n_1}, \ldots, \omega_{n_N} \) be \( N \) real numbers. Let us denote by \( T = (n_1^{a_1}, \ldots, n_N^{a_N}) \), \( a_i = 0, 1, 2, \ldots \), a subset of \( A \) with multiple occupations possible: the \( a_i \)'s denote the multiplicity of the site \( i \).

We define \( \mathcal{T} = \) base of \( T = \) set of the occupied points in \( T = (n_1^{a_1}, \ldots, n_N^{a_N}) \) and \( |T| = \sum a_i \); put

\[
\omega(T) = \prod_{i=1}^{N} \omega_{n_i}^{a_i}.
\]

Let the function \( q_0 \) be defined on \( R^d \) as

\[
q_0(x) = \begin{cases} 
\omega_{n_i} & \text{if } x \text{ belongs to the unit square around } n_i, \\
0 & \text{otherwise.}
\end{cases}
\]

Then (3.8) becomes the characteristic function (1.3) for the block spins \( \{v_n\}_{n \in A} \)

\[
\langle \exp[iv(\varphi)] \rangle = \langle \exp \left[ i \sum_{n \in A} \omega_n v_n \right] \rangle,
\]

and this value can be written as

\[
\exp \left[ \sum_{n \equiv \ell \leq k} i^{\ell(2)} J^{(2)}_x \omega(T) + \theta L^{-d(k-\frac{d+1}{2})} B_\varphi M_\varphi D_k \right],
\]

where, if \( T = (n_1^{a_1}, \ldots, n_N^{a_N}) \) and \( T! = a_1! \cdots a_n! \),

\[
J^{(2)}_x = \sum u_{\ell \equiv \ell \leq k} (\xi_1, \ldots, \xi_{|T|}) L^{-d(3d-2)} \cdot (T!)^{-1},
\]

and the sum runs over the \((\xi_1, \ldots, \xi_{|T|})\), such that \( a_1 \) of them lie inside the box \((n_1, L)\) (cf. the Introduction for this symbol), \( a_2 \) lie inside \((n_2, L)\) etc.; the variable \( q \) has been introduced in (3.14) for the purposes of the following discussion but is, here, to be taken equal to 1.

3) The sum in (3.13) could be restricted to the \( T \)'s which involve only nearest-neighbour sites, i.e. to \( T \)'s with base \( T \) contained in a unit cell of the
lattice. This is a consequence of the exponential-decay property mentioned in Theorem 1. Of course the constant \( D_h \) would have to be changed to a new value \( \tilde{D}_h \).

4) There is a very simple relation between the constants \( J_x^{(a^*a)} \) and \( J_x^{(a^{*a*})} \) and it is worth remarking:

\[
J_x^{(a^{*a*})} = 2^{-\delta q^2} \sum_{k \in \mathcal{R}} J_x^{(a^*a)},
\]

where the operation of dividing a set \( R = (n_1^a, \ldots, n_p^a) \) by 2 means the following (see Fig. 1): think of \( Z^d \) as the union of adjacent disjoint unit cells and identify all the \( 2^d \) points of a unit cell with a single point of a new lattice \( Z^d \) (in the natural way): in this process the points of the set \( R \) become points of the new lattice; we attribute to each of them a multiplicity equal to the sum of the multiplicities of the points that are «contracted into it»: the set we obtain is \( T = \frac{1}{2} R \).

We call the above transformation the linear renormalization group for the block spins.

5) Let us note the following formal series expansions for \( \langle \exp [i \omega v_q(q^0)] \rangle \) and \( \langle \exp [i \omega \nu_{q_2}] \rangle \), where \( v_q(q^0) \), \( \nu_{q_2}(q^0) \) denote, respectively, the random variables (3.1) and (1.1) with the \( \omega \)-dependence explicitly marked:

\[
\langle \exp [i \omega v_q(q)] \rangle = \exp \left[ \sum_{p \geq 2} \frac{(i \omega)^p}{p!} \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} u_p(\xi_1, \ldots, \xi_p) \varphi \left( \frac{\xi_1}{L} \right) \ldots \varphi \left( \frac{\xi_p}{L} \right) \right],
\]

\[
\langle \exp [i \omega \nu_{q_2} \varphi] \rangle = \exp \left[ \sum_{p \geq 2} i^p J_2^{(p)} \omega^p \right],
\]

where the \( J \)'s are defined in (3.14).

The above series are asymptotic series in the sense of (3.8) and (3.13) respectively.
6) The limit distribution of the block spins is a (trivial) fixed point for
the renormalization group (3.15) with $\phi = 1$

\begin{equation}
J_r^{(\omega)} = 0 \quad \text{except when} \quad T = \{\xi^z\} \quad \text{and} \quad J_{\xi^z}^{(\omega)} = \sum_{\xi} u_2(0, \xi).
\end{equation}

7) In the case $h = 0$ and under the assumption of exponential decay
of $u_3$ the above asymptotic formulae still make sense but one can really prove
them in a very weak sense: more precisely it is possible to prove, using Theo-
rem 2, that the joint probability distribution of the block spins $\{v_n\}_{n \in \mathbb{N}}$ converges
weakly to a product of independent Gaussians with dispersion (3.18).

This statement simply means that the mean central-limit theorem holds
for $v(r)$ in this case too: its proof is essentially a repetition of the proof in (10),
once Theorem 2 is taken into account. Therefore we do not give the details.

4. – About the critical point.

In this Section we shall consider the consequences of various subsets of the
set of audacious assumptions listed below. The subscript $c$ will mean that the
quantity we are considering is computed at $\beta = \beta_c, \quad h = 0$. If we consider some
quantity for the values $(\beta, h)$ of the inverse temperature and the field, we shall
write those variables in parentheses after the symbol. For a definition of
$\beta_c$ see (11):

I) \begin{equation}
\langle \sigma_0 \sigma_r \rangle_c = \frac{a_0}{|r|^{d+n\eta - 2}} \left(1 + O\left(\frac{1}{|r|}\right)\right)
\end{equation}
for some $\eta, \quad 2 > \eta > 0$, and $a_0 > 0$;

II) for some $m_0 > 0$ and $\delta > 1$

\begin{equation}
\eta_1(\xi)_{(h, \rho, h)} = M(h) = m_0 h^{\delta\rho}(1 + O(h));
\end{equation}

III)

\begin{equation}
\frac{d^n}{dh^n} M(h) = m_0 \frac{1}{\delta} \left(\frac{1}{\delta} - 1\right) \ldots \left(\frac{1}{\delta} - n + 1\right) h^{(\delta\rho - n)}(1 + O(h))
\end{equation}
for $h \to 0^+$;


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IV) if $\Lambda(\xi_1, ..., \xi_n)$ is the length defined in Theorem 1

\begin{equation}
\sigma_{2n}(0, r_1, ..., r_{2n-1}|\sigma, n) \propto \exp \left[ - \frac{L(h)}{\sigma} \Lambda(0, r_1, ..., r_{2n-1}) \right] \sigma_{2n}(0, r_1, ..., r_{2n-1}) \sigma,
\end{equation}

where $\propto$ means that the sums of the two sides over $r_1, ..., r_{2n-1}$, agree to leading order in $h$ within $O_n(h)$, if $L(h)$ is properly chosen (see V) below);

V)

\begin{equation}
L(h) = l_o h^{-\upsilon} (1 + O(h)) \quad \text{for some } l_o, \upsilon > 0;
\end{equation}

VI) $\sigma_{2n}(0, r_1, ..., r_{2n-1})$ is asymptotic to a homogeneous function $\chi_{2n}(0, r_1, ..., r_{2n-1})$ of degree $\omega_{2n}$, i.e. such that

\begin{equation}
\chi_{2n} \left( 0; \frac{r_1}{\sigma}, ..., \frac{r_{2n-1}}{\sigma} \right) = \sigma^{\omega_{2n}} \chi_{2n}(0, r_1, ..., r_{2n-1})
\end{equation}

in the sense that the sum of $\sigma_{2n}$ over $r_1, ..., r_{2n-1}$ such that $|r_i| < R$ and the same sum with $\chi_{2n}$ instead of $\sigma_{2n}$ agree to leading order in $R$ as $R \to \infty$ within $O(1/R)$. Clearly in the above statements $O(x)$ means

$$\limsup_{x \to \infty} \frac{|O(x)|}{x} < \infty.$$ 

Some of the results below will be obtained using assumption III), IV), VI) only for some values of $n$.

Assumption I) can be explicitly verified in only one interesting instance: see Theorem 3. Assumptions II), III), V) have never been rigorously verified in the class of models we are considering, but II) and V) appear often in the literature in a form similar or equal to the one we are adopting here (see for instance (4)). Assumption IV) is the most audacious and has not only never been verified rigorously in any model but, at least to our knowledge, does not seem to have ever been considered in the literature. IV) is a very strict interpretation of the following loose but very frequent statements which are basic ingredients for many theories of the scaling laws: « there is only one correlation length near $\beta = \beta_c$, $h = 0$ » and « within the correlation length the spins are at criticality ». Of course IV) is not the only possible interpretation of the above phrases, which, on the other hand, are usually made in connection with the single spin and pair correlation functions only.

The way the correlation length is introduced in IV) is inspired by recent rigorous results on the cluster property in the Ising model (12) in the $h \neq 0$ region or in the high-temperature region.

Finally we shall use the fact that, if \( h = 0 \), \( \beta < \beta_c \), then (6)

\[
u_2 > 0, \quad \nu_4 < 0.
\]

Let us now investigate the consequences of the above assumptions.

\[ a) \text{ If } I) \text{ is assumed, the criterion } (1.4) \text{ for the choice of } \varphi \text{ gives at } h = 0 \]

\[
(4.7) \quad \varphi = 1 + \frac{2 - \eta}{d}.
\]

\[ b) \text{ Assumption I) allows one to show that the first term in the expansion } (3.16) \text{ for } \langle \exp [itv(q)] \rangle \text{ has a limit when } L \to \infty \text{ and one finds that this limit is} \]

\[
(4.8) \quad - \frac{t^2}{2} a_0 \int \! dx \, dy \, \frac{q(x) q(y)}{|x - y|^{d-\eta}},
\]

which corresponds to a block-spin pair coupling (see (3.14)) \( J_{\eta,m} \) which tends, as \( L \to \infty \), to

\[
(4.9) \quad J_{\eta,m}^{(o)} = - \frac{a_0}{2} \int \sum_{n \in [n]} \sum_{m \in [m]} |x - y|^{2-d-\eta},
\]

where \([n]\) = unit cube with centre \( n \).

It is easy to check that, as expected, \( J_{\eta,m}^{(o)} \) is a fixed point for the renormalization group (3.15) for \(|T| = 2\).

The rest of this Section is devoted to the higher-order terms in (3.16), (3.17).

\[ c) \text{ Assume I)-V) for } n = 1 \text{ only. Then using the well-known formula} \]

\[
(4.10) \quad \frac{dM(h, \beta_c)}{dh} = \beta_c \sum_{r} u_{2}(0, r)(\nu, \delta), \quad h > 0,
\]

and comparing the result with II), we find a relation between \( \bar{v}, \delta, \eta \). Notice that IV) and I) coincide in this case. More precisely we find

\[
(4.11) \quad \bar{v} = \frac{\delta - 1}{2} \frac{1}{2 - \eta},
\]

and also

\[
(4.12) \quad \mu_{-2-\eta} = \frac{a_0 \beta_c}{m_0} \int \! \exp[-\Lambda(0, x)] |x|^{2-\eta-\delta} \, dx.
\]
d) Assume I)-VI) for all $n$. Then it is easy, proceeding as above, to derive a relation between $\omega_{2n}$, $\delta$, $\eta$ and express $\omega_n$ as a function of $\delta$, $\eta$. We find

\begin{equation}
\bar{\nu}(2n+1)d - \omega_{2n+2} = 2n + 1 - \frac{1}{\delta},
\end{equation}

and also

\begin{equation}
m_0(2n+1)! \left( \frac{1/\delta}{2n+1} \right) = \beta^{2n+1} \frac{\lambda(2n+1)(\delta-1)\delta^{-1}}{2^{2n+1}} I_{2n+1},
\end{equation}

where

\begin{equation}
I_{2n+1} = \int \exp \left[ -A(0, x_1, \ldots, x_{2n+1}) \right] \chi_{2n+1}(0, x_1, \ldots, x_{2n+1}) \, dx_1 \ldots dx_{2n+1}.
\end{equation}

We can use (4.13), (4.14) and I)-VI) to find an asymptotic form for the $p$-th term of the formal series (3.16). We find, with $u_{2n+1} = 0$ and using (4.13), (4.11), (4.7),

\begin{equation}
\sum_{p\geq 1} \frac{i^p t^p}{(2p)!} L^{-(p-1)((\delta+1)/(\delta-1))(-\eta+1)} \int dx_1 \ldots dx_{2p} \chi_{2p}(x_1, \ldots, x_{2p}) \varphi(x_1) \ldots \varphi(x_{2p}),
\end{equation}

which has to be interpreted in the sense that the $p$-th term of this series is the leading asymptotic form of the corresponding term in (3.16). So we see that if the scaling law (see Theorem 4)

\begin{equation}
\frac{\delta + 1}{\delta - 1} (2 - \eta) - d > 0
\end{equation}

holds with the $>$ sign, then all but the first term in (4.16) vanish as $L \to \infty$ and we formally (but not rigorously) find that the asymptotic distribution of $v_\nu(p)$ and, hence, of the block spins is a Gaussian. If the scaling law (4.17) holds with the equality sign, then we find that all terms in the formal series for $\langle \exp [itv_\nu(p)] \rangle$ are important as $L \to \infty$ and, again only formally,

\begin{equation}
\langle \exp [itv_\nu(p)] \rangle_\varepsilon \to \exp \left[ \sum_{p\geq 1} \frac{(-1)^p}{(2p)!} t^{2p} \int dx_1 \ldots dx_{2p} \chi_{2p}(x_1, \ldots, x_{2p}) \varphi(x_1) \ldots \varphi(x_{2p}) \right].
\end{equation}

e) Now suppose that the scaling law (4.17) holds with the equality sign. Then formula (4.18) implies that the characteristic function for the block spin $v_{2,0}$ is given, in the limit $L \to \infty$, and in a formal sense, by

\begin{equation}
\exp \left[ -\sum_{p\geq 1} \frac{(-1)^{p+1}}{(2p)!} \omega^{2p} \int \chi_{2p}(x_1, \ldots, x_{2p}) \, dx_1 \ldots dx_{2p} \right],
\end{equation}

where $[0]$ is the unit cube with centre 0.
The integral in (4.19) resembles very much (4.15) and if one assumes

\[
\text{sign } u_{zn} = (-1)^{n+1},
\]

which is rigorously true \((\ast)\) for \(n=1, 2\) and is consistent with \(\Pi\) for all \(n\), then one can use the expression (4.14) or (4.15) to put upper and lower bounds on the integral in (4.19): we get

\[
0 \leq (-1)^{n+1} \int_{\Omega_1} \chi_{2n}(x_1, \ldots, x_{2n}) \, dx_1 \ldots \, dx_{2n} \leq \frac{C m_0}{\ell^{(2p-1)d-1} d} (2p-1)! \left( \frac{1/\delta}{2p-1} \right) \beta^{-(2p-1)},
\]

where \(C > 0\) is a suitable \(d\)-dependent (but \(p\)-independent) constant. A lower bound of the same type can also be obtained.

The interest of the above inequality lies in the fact that it actually shows that the formal series (4.19) has a positive radius of convergence.

1) Formula (4.18) also provides an expression for the \(J^{(c)} = \lim_{L \to \infty} J^{(c)}_n\) in terms of the sequence of homogeneous functions \(\{\chi_m\}\).

It is easy to check that \(J^{(c)}_n\) is, indeed, a fixed point of the linear renormalization transformation (3.15) with \(\rho\) given by (4.7). Notice that this fact only depends upon the homogeneity of \(\chi_{zn}\) with a homogeneity degree \(\omega_{zn}\) verifying (4.13) with \(\bar{\varphi}\) given by (4.11), with \(\delta\) and \(\bar{\varphi}\) related by (4.17) (with the equal sign); \(\eta\) is arbitrary.

So we have a large class of fixed points of (3.15).

5. - Conclusions.

The results of this paper support the conjecture that for all values \((\beta, h)\), and with the appropriate choice of the scaling parameter \(\rho\), the block-spin distributions have a limit as \(L \to \infty\) and this limit is a fixed point of the linear renormalization group (3.15).

The critical point \((\beta_c, 0)\) should be characterized as the only point \((\beta, h)\) for which \(\rho \neq 1\) (i.e. the central-limit theorem fails to hold).

If the parameter \(\rho\) which is correct for \((\beta_c, 0)\) is used for \((\beta, h) \neq (\beta_c, 0)\), then the block-spin distributions should have a trivial limit: namely the block spins should all become zero with probability 1. This event should manifest itself through coefficients \(J^{(c)}_n\) which all become zero as \(L \to \infty\).

We have been able to prove rigorously the above picture in a rather strong sense in the noncritical case \(h \neq 0\) and a weak version of it in the case when \(h = 0\) and \(u_2(\xi, \xi')\) decays exponentially (Sect. 3).
For heuristic purposes we have made a number of assumptions about the behaviour of the correlation functions at the critical point. These assumptions allow a discussion of the case $\beta = \beta_c$, $h = 0$ (Sect. 4).

Under the assumptions of Sect. 4 we show that it is indeed possible to choose $\varrho$ in such a way that the formal series for the logarithm of the characteristic function of the block-spin distribution (3.16), (3.17) is term-by-term convergent to a limit formal series when $L \to \infty$.

We show that this limit series either contains only one term, the quadratic one, or contains all the terms and these two events depend, respectively, on whether the scaling law

$$2 - \eta \leq \frac{\delta - 1}{\delta + 1}$$

holds with the $<$ sign or with the $=$ sign (this law is a consequence of the assumptions).

In the case of the equality sign in (4.1) one can see that the formal series has a nonvanishing (though finite) radius of convergence and the parameters $J_x$ associated with it (and describing the Fourier transform of the block-spin distributions) are a solution of the renormalization group equations (3.15).

We wish to conclude this paper trying to make a connection with Wilson's theory of the critical point.

One obtains the Kadanoff-Wilson renormalization group as follows: consider the joint probability distribution of the block spins, for given $L$, and find a formal Hamiltonian whose Gibbs distribution gives rise to them. As $L$ changes we obtain a sequence of Hamiltonians and the transformation which relates them is the renormalization transformation introduced by Kadanoff and to which Wilson's theory applies.

One basic conjecture in Wilson's theory is that at $\beta = \beta_c$, $h = 0$ and if $\varrho$ is properly chosen this sequence of effective Hamiltonians has, in some sense, a limit which, of course, should be invariant under further renormalizations.

It is clear that the analysis of this paper deals with a similar question; we point out the rather obvious fact that with an appropriate choice of $\varrho$ one should « always » approach a limiting distribution and prove the statement for all $h \neq 0$ and also for some $(\beta, 0)$. Under the assumptions of Sect. 4 we show that this should still be true at $\beta = \beta_c$, $h = 0$ and notice a number of relations of this problem with the scaling law (4.1). We stress that the assumptions I)-VII) of Sect. 4 are either true or false for the Ising model and we are unable

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to decide this; furthermore, even if the assumptions I)-VII) could be proven, our proof would still be of heuristic character because it deals with the term-by-term study of a series and not with its sum. The other fundamental assumption in Wilson's theory is that, if \( \beta = \beta_c \) (say) and \( h \sim 0 \), then the above-defined effective Hamiltonian for the block spins is, for \( L = L_0(h) \ll L(h) = \text{correlation length} \), very close to the invariant Hamiltonian associated with \( \beta = \beta_c, \ h = 0 \), and the difference between the two is, in some sense, proportional to \( h \).

This assumption looks after an attentive examination quite reasonable. Clearly the results of this paper have something to do with the problem of showing that the above statement is in some sense true: formula (3.13) gives an expression, rigorous, for the block-spin distributions for any \( h \neq 0 \) and (4.18) gives a nonrigorous form for the same quantities at \( \beta = \beta_c, \ h = 0 \): so our results go in the direction of studying the relation between the Fourier transforms of the block-spin distributions at \( \beta = \beta_c, \ h \neq 0 \) and \( \beta = \beta_c, \ h = 0 \). However there seems to be no reason to believe that these objects will show the same regular behaviour in \( h \) as the effective Hamiltonian and one may imagine the departures from the critical behaviour to be of order \( h^\lambda \) with \( \lambda \neq 1 \). The trouble lies in the fact that the transformation from the distributions to the corresponding effective Hamiltonians is expected to behave quite singularly in the neighbourhood of a distribution which is itself singular, so this transformation may add singularities. This is the reason why one should not yield to the strong temptation of applying to the above renormalization group for the distributions of the block spins the same ideas Wilson used in connection with the Kadanoff renormalization group.

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APPENDIX

C. PFISTER

ETH, Theoretische Physik, Zurich

Proof of Theorem 1.

The proof follows the lines of paper (14).

Let \( \xi_\ell \) be a function defined for \( \xi \in M = \{\xi_1, \ldots, \xi_\ell\} \subset Z^d \) and such that

Consider the function

$$\hat{Z}(\zeta) = \hat{Z}(\zeta_1, \ldots, \zeta_n) = \sum \exp \left[ \beta \sum \frac{J(\xi - \xi') \sigma_\xi \sigma_{\xi'} I}{\xi} \right] \prod \zeta_\xi^{1-\sigma_\xi}.$$  

It is clear that

$$u_1^{(\xi)}(\zeta_1) = \frac{\hat{Z}(i\xi_1, \zeta, \ldots, \zeta_n)}{\hat{Z}(\zeta_1, \zeta, \ldots, \zeta_n)} = -\frac{\partial \log \left( \prod \zeta_\xi \right) Z(\zeta)}{\partial \log \zeta_1},$$

and also

$$\hat{Z}(\zeta_1, \ldots, \zeta_n) = \zeta_1^p P(\zeta') + Q(\zeta'),$$

where $P, Q$ are polynomials in the variables $\zeta_\xi$ with $\xi' \neq \xi_1$.

Since $Z$ does not vanish in the domain under consideration, we must have

$$\min_{i\xi_1 < R} \left| \frac{Q(\zeta')}{P(\zeta')} \right| > R^2.$$

But

$$u_1^{(\xi)}(\zeta_1) = 1 - \frac{2\zeta_1}{\zeta_1 + Q(\zeta') P(\zeta')}$$

so that, if $|\zeta_1| < R(1 - \varepsilon), R > \frac{1}{2}$ and $\varepsilon$ small enough, we find

$$|u_1^{(\xi)}(\zeta)| \leq \frac{2}{\varepsilon} \quad \text{if} \quad |\zeta_1| < R - \varepsilon, \quad \forall \xi < M.$$

Let us write $u_1^{(M)}(\zeta_1, \ldots, \zeta_n)$ instead of $u_1^{(\xi)}(\zeta_1)$ to make more explicit the $\zeta$-dependence; then Morera's theorem allows us to write for $|\zeta_1| < R - \varepsilon$

$$u_1^{(M)}(\zeta_1, \zeta, \ldots, \zeta_n) = (2\pi i)^{-k} \oint \frac{dz_1 \ldots dz_k}{(z_1 - \zeta_1) \ldots (z_k - \zeta_k)} \cdot u_1^{(\xi)}(\zeta_1, \ldots, \zeta_k, \zeta, \ldots, \zeta_n),$$

where the $\oint$ runs over a circle of radius $R - \varepsilon$.

Clearly (A.7), together with (A.6) and (2.2), implies that, in the notation of Sect. 2,

$$|u_1^{(M)}(\zeta_1, \ldots, \zeta_n)| < B_{\zeta_1, \zeta},$$

which holds for an arbitrary choice of $\xi_1, \ldots, \xi_n$ in $M$, since the initial labelling was arbitrary.

Next we introduce a complex parameter $t, |t| < 1$, and consider the function

$$u_k^{(M)}(\zeta_1, \ldots, \zeta_k|\zeta),$$

where $\zeta = t\zeta_1$.

If $|\zeta_1| < R - \varepsilon$, then $u_k^{(M)}$ is a analytic function of $t$ in the unit circle, with obvious notations ($^{12}$)

$$u_k^{(M)}(\zeta_1, \ldots, \zeta_n|\zeta) = \varphi \sum_{\beta=0}^{\infty} C_{\beta}(\xi_1, \ldots, \xi_n|\zeta) t^\beta.$$
The bound (A.8) implies
\[(A.10)\]
\[|C_h^{(\nu)}(\xi_1, \ldots, \xi_k)| << B_{k,e},\]
and furthermore it is well known that \(C_h^{(\nu)}(\xi_1, \ldots, \xi_k)\) vanishes unless
\[(A.11)\]
\[\frac{A(\xi_1, \ldots, \xi_k)}{\lambda} + k - 1 < h,\]
where \(\lambda\) is the range of the interaction (see, for instance, (12), formulae (6), (7), (8), (9) where \(q^\beta(X, Y; \beta)\) has to be replaced by \((\prod_{\xi \in x} \zeta_\xi)(\prod_{\eta \in \eta} \zeta_\eta)q^\beta(X, Y; \beta)\) and \(z\) by \(t\).

Thus
\[(A.12)\]
\[|u_h^{(\nu)}(\xi_1, \ldots, \xi_k)| \leq t^k \sqrt{k^{1/2} \lambda^3} \frac{B_{k,e}}{1 - \sqrt{t}},\]
valid for \(\forall \xi, |\xi| < R - \varepsilon, \forall t < 1.\)

The bound (2.4) follows immediately if \(|\xi| < R - 2\varepsilon, \forall \xi\) (by choosing \(t = (R - 2\varepsilon)/(R + \varepsilon)\) and replacing, in (A.12), \(\xi\) by \(\xi/\lambda\)).

\begin{itemize}
  \item **RIASSUNTO**
\end{itemize}

Si derivano alcuni risultati rigorosi sulle distribuzioni dei blocchi di spin per \((\lambda, h) \neq (0, \beta_0)\). In base a certe ipotesi sulle funzioni di correlazione critiche si specula sulla natura delle distribuzioni dei blocchi di spin al punto critico.

\begin{itemize}
  \item Распределения спинов блоков для моделей Ишина с короткодействующим притяжением.
\end{itemize}

Резюме (*). — Мы приводим некоторые строгие результаты для распределений спинов блоков для \((\lambda, h) \neq (0, \beta_0)\). При определенных предположениях относительно функций критических корреляций мы рассматриваем распределения спинов блоков в критической точке.

(*) Переведено редакцией.