The Hierarchical Model and the Renormalization Group.

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1. – The model.

The models that we consider are 1-dimensional ferromagnets with a rather peculiar, nontranslationally invariant interaction [1-4]. It will be seen that the special form of this interaction leads to a rather simple formulation of the renormalization group equation. Apart from the lack of translation invariance these models are supposed to behave in many respects like ordinary 1-dimensional ferromagnets with (in the interesting case) a long-range interaction of strength $r^{-\alpha}$, $1<\alpha<2$.

Let $n = 1, 2, \ldots$ label the points of a semi-infinite one-dimensional lattice. On each site sits a spin σ_n which is a real number $\sigma_n \in (-\infty, \infty)$. For each integer p we break the lattice up in groups of 2^p consecutive sites; the m-th p-group will consist of the sites

$$(m-1)2^p + 1 \leqslant n \leqslant m2^p\,, \qquad m = 1, 2, \ldots. \tag{1.1}$$

We denote this group by the symbol (m, p).

It is convenient to introduce the magnetization of the p-group (m, p)

$$M_{(m,p)} \equiv \sum_{n\in(m,p)} \sigma_n\,. \tag{1.2}$$

Let $\Lambda_L = \{1, 2, ..., 2^L\}$, we define the energy of a spin configuration $\underline{\sigma} = (\sigma_1, ..., \sigma_{2^L})$ in Λ_L by

$$H_L(\underline{\sigma}) = -\sum_{p=1}^{L} D_p \sum_{m=1}^{2^{L-p}} M_{2m-1,p-1} M_{2m,p-1}, \tag{1.3}$$

in which the constants D_p are given by

$$D_p = D2^{-\alpha p}, \qquad \alpha > 1. \tag{1.4}$$

This Hamiltonian will be called the hierarchical Hamiltonian; other versions of essentially the same Hamiltonian will be introduced later.

The definition of the model is completed by giving a normalized, even probability distribution $\pi(\sigma)$, which will be called the «free-spin distribution» and which describes the probability of finding an uncoupled spin of length σ. The partition function of the model defined by the couple $\{H_L, \pi\}$ is defined as

$$Z_L(\pi, \beta) = \int \prod_{i=1}^{2^L} \mathrm{d}\sigma_i \, \pi(\sigma_i) \exp[-\beta H_L(\underline{\sigma})]. \tag{1.5}$$

We will always assume that the free-spin distribution is such that this integral makes sense. Since H_L is a quadratic form with a lowest eigenvalue given by

$$\lambda_{\min} = -\frac{\frac{1}{4}D(1-2^{(1-\alpha)L})}{2^{\alpha-1}-1}, \tag{1.6}$$

distributions which fall off like

$$\pi(\sigma) \simeq \exp\left[-\left(\frac{\frac{1}{4}D\beta}{2^{\alpha-1}-1}\right)\sigma^2\right] \tag{1.7}$$

are just admissible. Any distribution which falls off faster, and in particular the distributions with compact support, are admissible. The models for which

$$\pi(\sigma) = \tfrac{1}{2}\delta(\sigma+1) + \tfrac{1}{2}\delta(\sigma-1) \tag{1.8}$$

will be called Ising-hierarchical models.

It will be convenient to describe the system in terms of the distribution of the magnetization $M_{(m,p)}$ of the p-groups. Since the Hamiltonian is symmetric in the number m of a p-group it suffices to introduce the distribution of $M_{(1,p)}$. Since one expects that this distribution will spread out for large p it is useful to rescale it with a factor ξ^{-p} and to define

$$\pi_{p,\xi}^{(L)}(v) = Z_L^{-1}(\pi, \beta) \int \delta(v - M_{1,p}\xi^{-p}) \exp[-\beta H_L(\underline{\sigma})] \prod_{i=1}^{2^L} \pi(\sigma_i)\,\mathrm{d}\sigma_i. \tag{1.9}$$

Normal dispersion corresponds to the value $\xi = \sqrt{2}$.

2. – Conjectures.

If the above model behaves like an «honest» ferromagnet one should expect the following behaviour. There should exist a β_c such that for $\beta<\beta_c$ the magnetization of large groups of spins has a normal dispersion. More precisely

$$\lim_{L\to\infty}\int_{v_1}^{v_2}\pi_{L,\sqrt{2}}^{(L)}(v)\,\mathrm{d}v = \left(2\pi\chi_l(\beta)\right)^{-\frac{1}{2}}\int_{v_1}^{v_2}\exp\left[-v^2/2\chi_l(\beta)\right]\mathrm{d}v\,, \tag{2.1}$$

and

$$\lim_{p\to\infty}\lim_{L\to\infty}\int_{v_1}^{v_2}\pi_{p,\sqrt{2}}^{(L)}(v)\,\mathrm{d}v = \left(2\pi\chi_s(\beta)\right)^{-\frac{1}{2}}\int_{v_1}^{v_2}\exp\left[-v^2/2\chi_s(\beta)\right]\mathrm{d}v \tag{2.2}$$

for all v_1, v_2. The convergence is not assumed to be pointwise to take into account the Ising case where $\pi_p^{(L)}$ is a sum of δ-functions.

The distribution corresponding to (2.1) will be called the long–long-order distribution whereas the distribution corresponding to (2.2) will be referred to as the short–long-order distribution.

At $\beta=\beta_c$ the dispersion is no longer normal and one expects that there exist constants ξ_l and ξ_s larger than $\sqrt{2}$ such that

$$\lim_{L\to\infty}\int_{v_1}^{v_2}\pi_{L,\xi_l}^{(L)}(v)\,\mathrm{d}v = \int_{v_1}^{v_2}f_l(v)\,\mathrm{d}v \tag{2.3}$$

and

$$\lim_{p\to\infty}\lim_{L\to\infty}\int_{v_1}^{v_2}\pi_{p,\xi_s}^{(L)}(v)\,\mathrm{d}v = \int_{v_1}^{v_2}f_s(v)\,\mathrm{d}v \tag{2.4}$$

(for all v_1, v_2) in which f_l and f_s are normalized distributions with a finite second moment.

It is usually assumed in dealing with the normal Ising model that the functions f_l and f_s and the values ξ_l, ξ_s and χ_l, χ_s are all equal. However, such statements are often not proved. We shall keep a different notation for the short–long-order and long–long-order quantities because later we shall become involved in a discussion on the possible differences.

3. – Existence of a phase transition for $1<\alpha<2$.

The coupling in the hierarchical model is of a ferromagnetic nature; in fact is easy to verify that the interaction is of the two-body type and can be written

$$H_L(\underline{\sigma}) = -\sum_{i,j}^{1,2^L} J_{i,j}\,\sigma_i\,\sigma_j \tag{3.1}$$

with

$$J_{i,j} = D 2^{-\alpha(d_{i,j}+1)}, \tag{3.2}$$

where $d_{i,j}$ is the largest integer such that i and j are not in the same d-group. Note that this definition implies

$$|i-j| < 2^{d_{i,j}+1}. \tag{3.3}$$

By adding ferromagnetic interactions we can therefore reach from the hierarchical model a model with a translation-invariant interaction of the form

$$\tilde{J}_{i,j} = D|i-j|^{-\alpha}. \tag{3.4}$$

We will only consider models in which $\pi(\sigma)$ is of the « Griffiths » type [5], which means that either $\pi(\sigma)$ can be written as

$$\pi(\sigma) = Z_N^{-1} \sum_{t_1,\dots,t_N}^{\pm 1} \exp\left[\sum_{i,j}^{1,N} J_{i,j} t_i t_j\right] \delta\left(N^{-a} \sum_{i=1}^{N} t_i - \sigma\right) \tag{3.5}$$

with $J_{i,j} \geqslant 0$ and Z_N a normalization factor, or $\pi(\sigma)$ is a weak limit of distributions $\pi_n(\sigma)$ of the form (3.4) $\Big($*i.e.*

$$\lim_{n\to\infty} \int_{v_1}^{v_2} \pi_n(\sigma)\, d\sigma = \int_{v_1}^{v_2} \pi(\sigma)\, d\sigma \qquad \text{for all } v_1,\, v_2\Big)$$

such that there exists a constant C with

$$\pi_n(\sigma) \leqslant C \exp\left[-\left(\frac{\frac{1}{4} D\beta}{2^{\alpha-1} - 1}\right)\sigma^2\right]. \tag{3.6}$$

The interest of such π's resides in the fact that ferromagnetic-spin models with a free-spin distribution of the Griffith's type (and a Hamiltonian which behaves for large σ's as in the present case) verify the GKS inequalities [6].

In particular, therefore, the existence of long-range order in the hierarchical model implies also long-range order in the corresponding translationally invariant model defined by (3.4). Notice that, by the results of [7], the presence of long-range order is equivalent to the presence of spontaneous magnetization at least in the case of a translationally invariant model with discrete spins.

Another Griffith's inequality that we shall use is

$$\langle \sigma_i \sigma_j \rangle_{L'} \geqslant \langle \sigma_i \sigma_j \rangle_L, \qquad L' \geqslant L, \tag{3.7}$$

where $\langle \cdot \rangle_L$ denotes the average with respect to the model $\{H_L, \pi\}$.

Let us now proceed to prove the existence of long-range order for large enough β in the hierarchical models ($1 < \alpha < 2$) and define

$$f_L(p) = 2^{-2p}\langle M_{p,1}^2\rangle_L = 2^{-2p}\sum_{i,j}^{1,2^p}\langle\sigma_i\sigma_j\rangle_L \tag{3.8}$$

and

$$f(p) = \lim_{L\to\infty} f_L(p)\,. \tag{3.9}$$

Notice that the above-mentioned Griffiths' inequality implies

$$f(p) \geqslant f_L(p) \geqslant f_p(p)\,. \tag{3.10}$$

So if we show that, for large enough β, $f_p(p) \geqslant C > 0$, we will have shown that

$$\limsup_{|i-j|\to\infty}\lim_{L\to\infty}\langle\sigma_i\sigma_j\rangle_L > 0\,, \tag{3.11}$$

i.e. the system exhibits (short) long-range order.

To study the function $f_p(p)$ write $M_{1,p-1} = s$, $M_{2,p-1} = t$; then

$$\begin{aligned} f_p(p) &= 2^{-2p}\langle(s+t)^2\rangle_p = 2.2^{-2p}\langle s^2\rangle_p + 2.2^{-2p}\langle st\rangle_p = \\ &= \tfrac{1}{2}f_p(p-1) + 2.2^{-2p}\langle st\rangle_p \geqslant \tfrac{1}{2}f_{p-1}(p-1) + 2^{-2p}\langle st\rangle_p\,. \end{aligned} \tag{3.12}$$

In terms of the distributions $\pi_{p,\xi}^{(L)}$ (defined in (1.9)) we have

$$\langle st\rangle_p = \frac{\int\pi_{p-1,1}^{(p-1)}(s)\pi_{p-1,1}^{(p-1)}(t)\exp[\beta D2^{-\alpha p}st]\,st\,\mathrm{d}s\,\mathrm{d}t}{\int\pi_{p-1,1}^{(p-1)}(s)\pi_{p-1,1}^{(p-1)}(t)\exp[\beta D2^{-\alpha p}st]\,\mathrm{d}s\,\mathrm{d}t}\,, \tag{3.13}$$

where we have taken advantage of the fact that the Hamiltonian couples spins in the groups $(p-1, 1)$ and $(p-1, 2)$ only through the product st.

We now make use of the following inequality [2] which holds for any random variable with an even distribution:

$$\frac{\langle x\exp[hx]\rangle}{\langle\exp[hx]\rangle} \geqslant \sqrt{\langle x^2\rangle}\,\mathrm{tgh}\left(h\sqrt{\langle x^2\rangle}\right), \tag{3.14}$$

which amounts to the estimate that the average of a continuous spin in a field h with respect to a distribution π decreases when this distribution is replaced by the sum of two delta-functions with the same value for $\langle x^2\rangle$. Application of this relation to the random variable st in (3.13) yields

$$\langle st\rangle_p \geqslant 2^{2(p-1)}f_{p-1}(p-1)\,\mathrm{tgh}\,[\beta D2^{(2-\alpha)p-2}f_{p-1}(p-1)]\,, \tag{3.15}$$

where we have used the definition (3.8) to evaluate the dispersion in st.

The estimate (3.12) takes now the form

$$f_p(p) \geqslant f_{p-1}(p-1)[1 + \exp[-\tfrac{1}{2}\beta D 2^{(2-\alpha)p} f_{p-1}(p-1)]]^{-1} . \tag{3.16}$$

Define now $f_0(0) \equiv \int \sigma^2 \pi(\sigma)\,\mathrm{d}\sigma \equiv 2C > 0$. Assume $f_n(n) \geqslant C$ for $n \leqslant p-1$, then it follows that

$$\ln f_p(p) \geqslant \ln C + \ln 2 - \sum_{n=1}^{p} \ln[1 + \exp[-\tfrac{1}{2}\beta D C 2^{(1-\alpha)p}]] . \tag{3.17}$$

Since this series is convergent for $\alpha < 2$ one can take β large enough so that the total sum does not exceed ln 2 and one can conclude $f_p(p) \geqslant C$ for all p by induction.

In order to prove the existence of a phase transition one has now still to show the absence of long-range order for high enough temperature. In the case of the Ising-hierarchical models this is a direct consequence of a result by GRIFFITHS [8] which states that an Ising model (with general two-body interaction) does not show long-range order for $T > T_0$, where T_0 is close to the mean field critical temperature and given by

$$\max_j \Big(\sum_{i \neq j} \mathrm{tgh}\,(J_{i,j}/kT_0)\Big) = 1 . \tag{3.18}$$

Notice that (3.2) implies indeed a finite value for T_0 provided $\alpha > 1$.

4. – The recursion formulae.

BAKER [3] noticed that the structure of the hierarchical model did particularly well fit the requirements for an exact formulation of the renormalization group equations [9-11], which in the case of the hierarchical model reduce to rather simple recursion formulae.

We will first discuss a recursion formula for the distribution function $\pi^{(L)}_{L,\xi}$ of the *total* magnetization of a system of size 2^L. Notice that the Hamiltonian H_L can be expressed recursively as

$$H_L(\sigma_1, \ldots, \sigma_{2^L}) = H_{L-1}(\sigma_1, \ldots, \sigma_{2^{L-1}}) + H_{L-1}(\sigma_{2^{L-1}+1}, \ldots, \sigma_{2^L}) - \\ - D 2^{-\alpha L} M_{1,L-1} M_{2,L-1} . \tag{4.1}$$

This relation implies directly

$$\pi^{(L)}_{L,\xi}(v) = \frac{1}{\text{normalization}} \pi^{(L-1)}_{L-1,\xi}(v_1) \pi^{(L-1)}_{L-1,\xi}(v_2) \exp[\beta D 2^{-\alpha L} \xi^{2(L-1)} v_1 v_2] \cdot \\ \cdot \delta(v_1 + v_2 - \xi v)\,\mathrm{d}v_1\,\mathrm{d}v_2 . \tag{4.2}$$

A privileged (but *a priori* arbitrary) choice for ξ is $\xi = 2^{\alpha/2}$, if we put $\pi^{(L)}_{L,2^{\alpha/2}} \equiv \pi^{(L)}_L$, then we find

$$\pi^{(L)}_L(v) = \frac{K\cdot \pi^{(L-1)}_{L-1}(v)}{\int K\cdot \pi^{(L-1)}_{L-1}(v)\,\mathrm{d}v}, \tag{4.3}$$

where the operator K is defined as

$$K\circ\pi(v) = \exp[C\zeta^2 v^2]\int_{-\infty}^{\infty}\mathrm{d}y\,\exp[-Cy^2]\pi(\zeta v + y)\pi(\zeta v - y), \tag{4.4}$$

with $C = \beta D 2^{-\alpha}$ and $\zeta = \frac{1}{2}\xi = 2^{\alpha/2-1}$. Other choices of ξ lead to an explicit L-dependence of the corresponding operator K.

There is a second recusion formula for the partition function which reflects more the idea of the contraction of variables in terms of the « block spins » first introduced by KADANOFF [12].

We start by observing the following alternative recursion formula for the Hamiltonian:

$$H_L(\sigma_1, \dots, \sigma_{2^L}) = H_{L-1}(\sigma'_1, \dots, \sigma'_{2^{L-1}}) - D2^{-\alpha}[\sigma_1\sigma_2 + \sigma_3\sigma_4 + \dots + \sigma_{2^L-1}\sigma_{2^L}], \tag{4.5}$$

where we defined the « block spins »

$$\sigma'_j \equiv 2^{-\alpha/2}(\sigma_{2j-1} + \sigma_{2j}). \tag{4.6}$$

The partition function is given by

$$Z_L(\pi, \beta) = \int \exp[-\beta H_{L-1}(\sigma'_1, \dots, \sigma'_{2^{L-1}}) + \beta D 2^{-\alpha}[\sigma_1\sigma_2 + \dots + \sigma_{2^L-1}\sigma_{2^L}]]\cdot \cdot\pi(\sigma_1)\dots\pi(\sigma_{2^L})\,\mathrm{d}\sigma_1\dots\mathrm{d}\sigma_{2^L}. \tag{4.7}$$

Integration over the internal variables of each block, keeping the « block spins » fixed, yields

$$Z_L(\pi, \beta) = (2^{\alpha/2})^{2^{L-1}}\int \exp[-\beta H_{L-1}(\sigma'_1, \dots, \sigma'_{2^{L-1}})]K\circ\pi(\sigma'_1)\dots K\circ\pi(\sigma'_{2^{L-1}})\cdot \cdot\mathrm{d}\sigma'_1\dots\mathrm{d}\sigma'_{2^{L-1}}, \tag{4.8}$$

where the operator K is defined in (4.4).

This formula expresses the partition function of a system of 2^L spins in terms of a hierarchical model with half that number of spins and with a new free-spin distribution $K\circ\pi$. The coupling constants in the Hamiltonian remain unchanged by virtue of the scaling chosen in (4.6).

The formula (4.8) can be written in the form

$$Z_L(\pi, \beta) = A^{2^{L-1}} Z_{L-1}(\pi', \beta), \tag{4.9}$$

where π' is the normalized distribution corresponding to $K\circ\pi$

$$\pi' = \frac{K\circ\pi(v)}{\int K\circ\pi(v)\,\mathrm{d}v}, \tag{4.10}$$

and the constant A is given by

$$A = 2^{\alpha/2}\int K\circ\pi(v)\,\mathrm{d}v. \tag{4.11}$$

Since clearly $\pi(v) \equiv \pi_0^{(0)}(v)$ we see from (4.3) that $\pi'(v)$ should be identified with the distribution $\pi_1^{(1)}$.

Iteration of the recursion process yields

$$\pi_L^{(L)}(v) = \frac{K^L\circ\pi(v)}{\int K^L\circ\pi(v)\,\mathrm{d}v} \tag{4.12}$$

and

$$Z_L(\pi,\beta) = 2^{(\alpha/2)(2^L-1)}\int_{-\infty}^{\infty} K^L\circ\pi(v')\,\mathrm{d}v'. \tag{4.13}$$

The formulae (4.4) and (4.9) are examples of the renormalization group equations as introduced in [9, 10].

We shall come back to this point later.

Finally we want to study the short-order distribution functions $\lim_{L\to\infty}\pi_{R,\xi}^{(L)}$. The recursion formula that we just derived applies only for the long-order distributions $\pi_{L,\xi}^{(L)}$. An expression for the short-order distributions can be obtained from the following recursive relation for the Hamiltonian:

$$H_L(\sigma_1, \ldots, \sigma_{2^L}) = H_R(\sigma_1, \ldots, \sigma_{2^R}) + \ldots + H_R(\sigma_{2^L-2^R+1}, \ldots, \sigma_{2^L}) + \\ + H_{L-R}(v_1, \ldots, v_{2^{L-R}}), \tag{4.14}$$

where $v_j \equiv \xi^{-R} M_{j,R}$ with $\xi = 2^{\alpha/2}$.

One obtains then for $\pi_{R,2^{\alpha/2}}^{(L)} \equiv \pi_R^{(L)}$

$$\pi_R^{(L)}(v) = \\ = \frac{1}{\text{normalization}}\int \exp[-\beta H_{L-R}]\,\delta(v-v_1)\pi_R^{(R)}(v_1)\ldots\pi_R^{(R)}(v_{2^{L-R}})\,\mathrm{d}v_1\ldots\mathrm{d}v_{2^{L-R}}. \tag{4.15}$$

The distributions for other values of ξ can be obtained from this relation by simple rescaling. The functions $\pi_R^{(R)}$ are the long-order distributions which satisfy the recursion relations (4.3).

We conclude that the short-order distributions are obtained by computing the *effective* single-spin distribution of a hierarchical model of infinite size where the successive iterates of the long-order distributions appear as the *free* single-spin distributions. We are interested in cases where the distribution functions tend to a well-behaved probability distribution with a scaling factor ξ chosen in such a way that the limiting distribution has a finite dispersion. There is no *a priori* reason to assume that such a ξ indeed exists. It is also not guaranteed, as we will see in an explicit example, that the same value of ξ will lead in the short-order and long-order cases to a limiting function with finite dispersion. We will come back to this point in our discussion of the influence of surface terms (see Sect. **8**).

5. – Nonrigorous considerations and Wilson's theory.

In this Section we proceed heuristically and try to give an exposition of the main ideas of Wilson's theory [9] as applied to the hierarchical model. In the preceding Section we derived a « simple » expression for $\pi_L^{(L)}$, namely

$$\pi_L^{(L)}(v) = \frac{K^L\circ\pi(v)}{\int K^L\circ\pi(v)\,\mathrm{d}v}\,, \tag{5.1}$$

where π is the initial free-spin distribution and K is the integral operator defined in (4.4). This operator depends explicitly on the temperature. It will be convenient to rescale in the sequel the variables of all distributions (with a common factor) so that K becomes temperature independent (say $\beta = 1$). The temperature dependence is then shifted to the initial distribution which becomes now a one-parameter family of initial free-spin distributions.

Let us further normalize the operator K and denote its normalized version by $\hat{K}$

$$\hat{K}\circ\pi_\beta(v) = \frac{\exp[C\zeta^2 v^2]\int\pi_\beta(\zeta v + y)\pi_\beta(\zeta v - y)\exp[-Cy^2]\,\mathrm{d}y}{\int\int\pi_\beta(\zeta v + y)\,\pi_\beta(\zeta v - y)\exp[C\zeta^2 v^2 - Cy^2]\,\mathrm{d}y\,\mathrm{d}v} \tag{5.2}$$

with $C = 2^{-\alpha}D$ and $\zeta = 2^{\alpha/2-1}$.

We are interested in the case that, for some $\beta = \beta_c$, $\hat{K}^L\circ\pi_{\beta_c}$ tends to a limiting probability distribution $\pi^{(\infty)}$ with a finite, nonvanishing dispersion. (It seems that, in order to be able to use the arguments that follow, one should require the pointwise convergence of the characteristic functions and of their first and second derivatives.)

This limit should then be a fixed point of the operator $\hat{K}$:

$$\pi^{(\infty)} = \hat{K}\circ\pi^{(\infty)}\,. \tag{5.3}$$

The natural question concerning the existence of fixed points for $\hat{K}$ is easily answered by observing that an obvious solution to this equation is the « Gaussian » fixed point

$$\pi_G^{(\infty)}(v) = \sqrt{a/\pi}\exp[-av^2] \tag{5.4}$$

with $a = \frac{1}{4}D/(2^{\alpha-1}-1)$. There may be many other fixed points for $\hat{K}$ in the above sense.

The existence of a limiting probability distribution with finite dispersion implies that the magnetization will have an average square value of the order of $2^{\alpha/2L}$, which is an anomalous dispersion (because $\alpha > 1$). Consequently the value β_c should be interpreted as a critical point of the system and we therefore expect it to be an isolated point, in the sense that for $\beta \simeq \beta_c$ the distribution $K^L \circ \pi_\beta$ will not have a good asymptotic behaviour. Since $K^L \circ \pi_\beta$ depends « smoothly » on β one expects on the other hand that for finite but large L_0 and $\beta - \beta_c$ small

$$\hat{K}^{L_0} \circ \pi_\beta(v) = \pi^{(\infty)}(v) + \psi_\beta(v) , \tag{5.5}$$

where ψ_β is « close » to zero and $\int \psi_\beta(v)\,\mathrm{d}v = 0$ due to normalization.

If π_β is an ordinary function (and not a general probability measure) the smoothness of the dependence on β necessary for (5.5) is obvious if one remembers (4.4) and the change of variables which transfers the β-dependence from the operator defined in (4.4) to the free-spin distribution. In the general case (which includes the interesting Ising model) eq. (5.5) has to be interpreted in a different way since, as we are assuming that $\pi^{(\infty)}$ is a smooth function, Ψ_β cannot be small in a trivial sense. The correct interpretation of (5.5) in the case $\pi_\beta(v)$ is a sum of delta-functions should essentially be that the mass of the points v which have nonzero measure with respect to $K^{L_0} \circ \pi_\beta$ will be proportional to $\pi^{(\infty)}(v) + \psi_\beta(v)$ with $\psi_\beta(v)$ smooth. In the rest of this Section we shall however assume that π_β is a smooth function and we leave to the reader the reformulation of the ideas of this Section in the Ising case using the above « interpolation » idea. We suggest, however, that the interested reader first read the next Section where a concrete example of the interpolation scheme is described in some detail from a rigorous point of view.

We proceed now to study what happens for $L > L_0$, and linearize the operator $\hat{K}$ around $\pi^{(\infty)}$. A direct computation yields

$$\hat{K} \circ (\pi^{(\infty)} + \psi_\beta) \simeq \pi^{(\infty)} + \hat{T}\,\psi_\beta , \tag{5.6}$$

where the linear operator $\hat{T}$ is defined as

$$\hat{T} \circ \psi = T \circ \psi - \pi^{(\infty)} \int T \circ \psi(v)\,\mathrm{d}v , \tag{5.7}$$

in which the operator T is given by

$$T \circ \psi(v) = 2\Lambda^{-1} \exp[C\zeta^2 v^2] \int \psi(\zeta v + y)\, \pi^{(\infty)}(\zeta v - y) \exp[-Cy^2]\,\mathrm{d}y , \tag{5.8}$$

and Λ is chosen such that $T \circ \pi^{(\infty)} = 2\pi^{(\infty)}$.

Notice that from the fact that $\pi^{(\infty)}$ is an eigenfunction (with eigenvalue 2) of the operator T, it follows that $\hat{T}$ has the same spectrum as the operator T, with the exception of the eigenvalue corresponding to $\pi^{(\infty)}$ which is zero for the operator $\hat{T}$.

To get some feelings about the operator T, consider it as a linear operator in the Hilbert space $L^2_{\text{even}}(\mathbf{R}, \exp[(2a-\gamma)v^2])$ of even functions which are square integrable with respect to the weight $\exp[(2a-\gamma)v^2]$ with

$$\gamma = \frac{D(1-\zeta^2)(\zeta^2-\frac{1}{4})}{\zeta^2(2\zeta^2-1)}. \tag{5.9}$$

(Notice that the allowed distributions are contained in this space.) The operator T is then a compact operator as can be seen by using the large-v behaviour of $\pi^{(\infty)}$.

If one assumes in particular that the model $\{H_L, \pi\}$ is such that at $\beta=\beta_c$ the distributions $\hat{K}^L\circ\pi_{\beta_c}$ tend to the Gaussian fixed point (5.4), then T becomes a self-adjoint compact operator T_G

$$T_G\circ\psi(v) = 2\sqrt{\gamma/(1-\zeta^2)\pi}\,\exp[-v^2 a]\cdot \cdot\int\exp[-(\gamma/1-\zeta^2)(\zeta v-y)^2+ay^2]\,\psi(y)\,\mathrm{d}y\,. \tag{5.10}$$

The Hermite functions

$$\psi_n(v) = \exp[-av^2]\,H_{2n}(v) \tag{5.11}$$

with

$$H_{2n}(v) = \exp[\gamma v^2]\frac{\mathrm{d}^{2n}\exp[-\gamma v^2]}{\mathrm{d}^{2n}v} \tag{5.12}$$

are eigenfunctions of T_G with eigenvalues given by

$$\lambda_n = 2\zeta^{2n}\,, \qquad n=0,1,\ldots. \tag{5.13}$$

The corresponding eigenvalues of $\hat{T}_G$ are found by replacing in this sequence $\lambda_0=2$ by zero. In case $2^{-\frac{1}{2}}<\zeta<2^{-\frac{1}{4}}$, which corresponds to $1<\alpha<\frac{3}{2}$, the operator $\hat{T}_G$ possesses one eigenvalue ($\lambda_1=2\zeta^2$) which is larger than one. As we will see, it is this situation that one expects to occur in the description of ordinary critical behaviour.

Let us assume for the moment that the operator $\hat{T}$ has in the general case only one eigenvalue larger than one (*) and express ψ_β in (5.5) in terms of the

(*) This, as it will become clear, is an essential assumption for the Wilson's theory of « ordinary » critical points (a different situation arises for critical points of higher order like tricritical points). More precisely it means that the spectrum of $\hat{T}$ in the linear eigenfunctions of $\hat{T}$

$$\psi_\beta(v) = \varepsilon_1(\beta)\,\psi_1(v) + \varepsilon_2(\beta)\,\psi_2(v) + \dots . \tag{5.14}$$

We obtain then by (5.6)

$$\hat{K}^{L_0+l}\circ\pi_\beta = \hat{K}^l\circ[\pi^\infty + \psi_\beta] \simeq \pi^{(\infty)} + \hat{T}^l\psi_\beta \simeq \tag{5.15}$$
$$\simeq \pi^{(\infty)} + \varepsilon_1(\beta)\,\lambda_1^l\,\psi_1 + \varepsilon_2(\beta)\,\lambda_2^l\,\psi_2 + \dots .$$

For $\beta = \beta_c$ this expression should tend to $\pi^{(\infty)}$ as $l \to \infty$; this implies $\varepsilon_1(\beta_c) = 0$. Should there have been another eigenvalue larger than one, then β_c would have satisfied several equations of this type at the same time so that only an accidental solution is possible. An exception seems the case where special symmetries of the system guarantee the absence of some terms in the eigenfunction expansion of ψ_β; notice however that this fact would be automatically taken into account by our definition of the domain of $\hat{T}$ and it would simply mean that *a priori* possible functions are, for symmetry reasons, absent from the domain of $\hat{T}$. For $\beta \simeq \beta_c$ we write

$$\varepsilon_1(\beta) \simeq e\cdot(\beta - \beta_c)\,, \tag{5.16}$$

and we see that $\hat{K}^{L_0+l}\circ\pi_\beta$ starts to deviate appreciably from $\pi^{(\infty)}$ as soon as $l > \ln(\beta - \beta_c)/\ln\lambda_1$ (*).

In the above context it is possible to find expressions for the critical exponents; we give some examples.

The exponent η describes the long-range behaviour of the spin-spin correlation function at the critical point through

$$\langle\sigma_i\sigma_j\rangle \simeq \frac{1}{|i-j|^{\eta-1}}\,. \tag{5.17}$$

This implies that the total magnetization of a subvolume of 2^L spins should diverge as

$$\langle M_{1,L}^2\rangle_L = \sum_{i,j}^{1,2^L}\langle\sigma_i\sigma_j\rangle \simeq 2^{L(3-\eta)}\,. \tag{5.18}$$

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space that is generated from the successive iterates $K^L\circ\pi$ has only a single eigenvalue larger than one.

In particular this means that the couple $\pi_G^{(\infty)}$ (Gaussian fixed point) and domain of $\hat{T} = L^2_{\text{even}}(\mathbf{R}, \exp[(2a-\gamma)v^2])$, hereafter often referred to as the « Gaussian case », cannot arise from the theory of an ordinary ferromagnetic hierarchical model if $\frac{3}{2} < \alpha < 2$.

(*) Notice that we are making use of the smoothness of the dependence of π_β on β. This might be a pecularity of the hierarchical model [13].

In the case of the hierarchical model the spin-spin correlation function is not translation invariant, consequently the relation (5.17) cannot be directly used, we take instead (5.18) as the relation defining η. As mentioned already the existence of a fixed-point $\pi^{(\infty)}$ with a finite width implies in view of the adopted scaling that

$$\langle M_{1,L}^2 \rangle_L \simeq 2^{\alpha L} . \tag{5.19}$$

One concludes therefore that

$$\eta = 3 - \alpha . \tag{5.20}$$

Another critical exponent is γ which describes the divergence of the susceptibility χ as $\beta \to \beta_c$ via

$$\chi(\beta) \simeq (\beta - \beta_c)^{-\gamma} . \tag{5.21}$$

The (long-order) susceptibility can be defined as

$$\chi(\beta) = \lim_{L\to\infty} \chi_L(\beta) \tag{5.22}$$

with

$$\chi_L(\beta) = 2^{-L} \langle M_{1,L}^2 \rangle_L = 2^{(\alpha-1)L} \int \hat{K}^L \cdot \pi_\beta(v)\, v^2 \, dv . \tag{5.23}$$

We now recall from the preceding discussion that for L_0 large enough and β close enough to β_c one has

$$\hat{K}^{L_0} \circ \pi_\beta \simeq \pi^{(\infty)} + e(\beta - \beta_c)\, \psi_1 + \varepsilon_2(\beta)\, \psi_2 + \dots$$

and

$$\hat{K}^{L_0+1} \circ \pi_\beta \simeq \pi^{(\infty)} + e(\beta - \beta_c)\, \lambda_1 \psi_1 + \varepsilon_2(\beta)\, \lambda_2 \psi_2 + \dots . \tag{5.24}$$

From this one obtains

$$\hat{K}^{L_0} \circ \pi_\beta \simeq \hat{K}^{L_0+1} \circ \pi_{\beta_c + (\beta - \beta_c)/\lambda_1} + \varepsilon_2' \psi_2 + \dots . \tag{5.25}$$

If the linear approximation was exact, a repeated application of $\hat{K}$ to this formula would make the terms containing the eigenvectors ψ_2, ψ_3, etc. vanish. We assume that this persists to be the case also for the nonlinearized equations. The conclusion then is

$$\lim_{L\to\infty} (\hat{K}^L \circ \pi_\beta - \hat{K}^{L+1} \circ \pi_{\beta_c + (\beta - \beta_c)/\lambda_1}) = 0 , \tag{5.26}$$

which should be valid asymptotically as $\beta - \beta_c \to 0$ in the real case. When we assume, more specifically, the equality in the limit $L \to \infty$ of the second moments, we find by the definition of the susceptibility the asymptotic expression (as $\beta \to \beta_c$)

$$\chi(\beta) \simeq 2^{(1-\alpha)} \chi(\beta_c + (\beta - \beta_c)/\lambda_1) , \tag{5.27}$$

which allows us to conclude this heuristic derivation with the value

$$\gamma = (\alpha - 1) \ln 2 / \ln \lambda_1 . \tag{5.28}$$

For the Gaussian fixed point, with $\lambda_1 = 2\zeta^2$, one finds in particular $\gamma = 1$, which is the classical or mean yield value.

As a third example we can study the correlation length ξ and its critical behaviour. We only study what should be called the long-order correlation length, *i.e.* the parameter that describes how large has to be the system's size L in order that the correlation between two spins σ_x and σ_y located at a distance 2^L starts depending on L as the interaction potential.

If $\beta \neq \beta_c$ one expects that the correlations verify Kadanoff's assumption that near β_c the spins have the same correlation they have at β_c provided their distance is smaller than the correlation length, while if their distance is much larger then their correlations are of the order of the interaction potential. This fact is clear from a heuristic point of view in the above renormalization group scheme: if we are close to β_c then $\pi_{L_0}^{(L_0)} = \hat{R}^{L_0} \circ \pi_\beta$ will be close to $\pi^{(\infty)}$, however, for $L = L_0 + l$, $\pi_L^{(L)}$ will substantially deviate from $\pi^{(\infty)}$ if $\beta \neq \beta_c$ and

$$\lambda_1^l (\beta - \beta_c) \sim 1 \tag{5.29}$$

(compare with (5.15)).

Therefore 2^l should be identified with the correlation length and we find

$$\xi(\beta) = 2^l \underset{\beta \to \beta_c}{\simeq} |\beta - \beta_c|^{-1/\log_2(\lambda_1)} = |\beta - \beta_c|^{-v} , \tag{5.30}$$

which means $v = 1/(\alpha - 1)$ for the corresponding critical exponent in the Gaussian case.

It is easy to verify that the critical exponents obtained so far satisfy the well-known scaling relation

$$\gamma = (2 - \eta) v . \tag{5.31}$$

The other critical exponents can be computed along similar lines and one can check that they obey the scaling laws.

So far we have only considered the long-range values of the critical exponents. The proper short-range value of η, η_s, should be obtained from the divergence of $\langle M^2_{1,R}\rangle_\infty$ as $R\to\infty$.

As we have seen in Sect. 4, one can write this as

$$2^{-\alpha R}\langle M^2_{1,R}\rangle_\infty = \lim_{L\to\infty}\langle v^2; K^R\circ\pi_{\beta_c}\rangle_L\,, \tag{5.32}$$

where the average of $v^2 = M^2_{1,0}$ is to be taken with respect to a model with free-spin distribution $\hat{K}^R\circ\pi_{\beta_c}$. This equation implies the same value for η_s and η_l provided that the r.h.s. of (5.32) remains finite as $R\to\infty$.

Suppose now that $\hat{K}^R\circ\pi_{\beta_c}$ tends for $R\to\infty$ to the Gaussian fixed point π_G^∞ and let us calculate $\lim_{L\to\infty}\langle v^2, \pi_G^\infty\rangle_L$.

At this point in our heuristic discussion we by-pass a study of the interchange of limits involved here.

Let us introduce a general Gaussian spin distribution by

$$\pi_r(v) = \exp[-(a+r)v^2]\,, \tag{5.33}$$

in particular one has $\pi_0 \propto \pi_G^{(\infty)}$. The average value of the random variable v^2 can then be obtained from

$$\langle v^2; \pi_G^{(\infty)}\rangle_L = -2^{-L}\frac{\partial}{\partial r}[\ln Z_L(\pi_r, \beta)]_{r=0}\,. \tag{5.34}$$

Recall now formula (4.13) to arrive at

$$\langle v^2; \pi_G^{(\infty)}\rangle_L = -2^{-L}\frac{\partial}{\partial r}\left[\ln\int_{-\infty}^{\infty} K^L\circ\pi_r(v')\,dv'\right]_{r=0}. \tag{5.35}$$

A direct calculation yields

$$K^L\circ\pi_r(v') = \left\{\prod_{l=0}^{L-1}[\pi/(2a+C+2\lambda_1^l r)]^{2^{L-l-2}}\right\}\exp[-(a+\lambda_1^L r)v'^2]\,. \tag{5.36}$$

Consequently we find

$$\langle v^2; \pi_G^{(\infty)}\rangle_L = \sum_{l=0}^{L-1}\frac{\frac{1}{2}(\lambda_1/2)^l}{2a+C} + \frac{\frac{1}{2}(\lambda_1/2)^L}{a} \tag{5.37}$$

with $C = D2^{-\alpha}$ as defined in (4.4) ($\beta = 1$).

Since $\lambda_1 = 2^{\alpha-1}$, one has $\lambda_1 < 2$ for $\alpha < 2$, which is the range of interest to us; in this case the second term vanishes for $L\to\infty$, while the first gives a finite limit

$$\langle v^2; \pi_G^{(\infty)}\rangle_\infty = 1/(2a+C)(2-\lambda_1)\,. \tag{5.38}$$

One therefore sees that the question of the identity of the short- and long-range values of η is linked to the above interchange of limits. Notice however that the amplitudes (*i.e.* the limiting distributions) do *not* coincide in the two cases.

The equality of the long-range and short-range exponents is a feature of the special hierarchical model that we are considering now. Later (in Sect. **8**) we shall encounter other versions of this model where not only the amplitudes but also the exponents differ in the two cases.

6. – The theorems of Bleher and Sinaii.

In the preceding Section we have seen that in case $1 < \alpha < \frac{3}{2}$ the Gaussian fixed point offers an example of the situation that one expects to occur in normal critical behaviour. The linearized version $\hat{T}_G$ of the operator $\hat{K}$ has the property that the fixed point $\pi_G^{(\infty)}$ is stable in the «critical hyperplane». This means that, whenever a small deviation ψ_{β_c} from the fixed point is contained in the hyperplane (of the appropriate Hilbert space) defined by $(\psi_{\beta_c}, \psi_1) = 0$, the repeated application of $\hat{T}_G$ yields $\hat{T}_G^l \circ \psi_{\beta_c} \to 0$.

What one would like to show now is that a similar situation arises for the nonlinear operator $\hat{K}$. More precisely one could hope to find a subset O_G of the hypersurface of «critical» free spin distributions surrounding $\pi_G^{(\infty)}$ such that $K^l \circ \pi \to \pi_G^{(\infty)}$ for $\pi \in O_G$.

The critical hyperplane, defined above, is expected to be «tangent» to O_G in $\pi_G^{(\infty)}$. For such π's the conjecture made in Sect. **2** about the (long-order) critical behaviour would be proven with $\xi_l = 2^{\alpha/2}$.

The important contribution of BLEHER and SINAII [4] is that they have proven the existence of a set O_G with the required properties. Since the shape of the critical surface is not known, their result is formulated in the following way. Let π_β for $\beta \in [\beta^-, \beta^+]$ be a one-parameter family of spin distributions which are close to $\pi_G^{(\infty)}$ in a way to be specified, then there exists a $\beta_c \in [\beta^-, \beta^+]$ such that $\hat{K}^l \circ \pi_\beta \to \pi_G^{(\infty)}$. The precise conditions on π_β are given in the following theorem.

Theorem 1. Let $1 < \alpha < \frac{3}{2}$. Let b, q and $\xi > q^{\frac{1}{4}}$ all be numbers between zero and one and let d be larger than one; the precise allowed values of these parameters depend only on α and are given in [4]. Let π_β be a family of spin distributions of the form

$$\pi_\beta(z) = L(\beta) \exp\left[-\big(a + b(\beta)\big) z^2\right]\big(1 + Q_\beta(z)\big) \tag{6.1}$$

with $L(\beta)$ and $b(\beta)$ differentiable and with the property that there exists an n_0 such that

(6.2) i) $|b(\beta)| \leqslant b(\beta^+)/\beta^+ = -b(\beta^-)/\beta^- = b^{n_0}$;

ii) if $|z| < d\sqrt{n_0/a}$,

then

$$Q_\beta(z) = \delta(\beta)\,H_4(z) + R(z, \beta)\,, \tag{6.3}$$

where H_4 is the Hermite polynomial defined in (5.12) and where

$$|R(z, \beta)| \leqslant q^{n_0}$$

and

$$(q/\xi)^{n_0} < -\,\delta(\beta) < (\sqrt{q}\,\xi)^{n_0}\,; \tag{6.4}$$

iii) if $|z| > d\sqrt{n_0/a}$ then

$$0 \leqslant 1 + Q_\beta(z) \leqslant \exp\,[-\,\mu z^4] \tag{6.5}$$

where $0 < \mu < (1/10)\,\delta(\beta)$.

If π_β satisfies these conditions, then there is an interval $[\beta_n^-, \beta_n^+] \subset [\beta^-, \beta^+]$ such that $\hat{K}^n \circ \pi_\beta$ satisfies the same conditions with n_0 replaced by $n_0 + n$.

Notice that the conditions on π_β are such that π_β cannot be too far from $\pi_G^{(\infty)}$ but moreover guarantees (especially by condition i)) that the curve described by π_β crosses the critical surface. Since $\pi_\beta \to \pi_G^{(\infty)}$ for $n_0 \to \infty$ the desired result is a direct consequence of this theorem.

Corollary. When

$$\beta_c = \bigcap_{n=1}^{\infty} [\beta_n^-, \beta_n^+] \qquad \text{then} \qquad \lim_{L\to\infty} \hat{K}^L \circ \pi_{\beta_c} = \pi_G^{(\infty)}\,. \tag{6.6}$$

We do not describe the proof of the above theorem, which is accomplished by a rather laborious induction.

The second theorem of BLEHER and SINAII is essentially identical to the first but allows the functions $\pi_\beta(z)$ to be sums of delta-functions. Assume we are given a family of functions $\bar{\pi}_\beta$, $\beta \in [\beta^-, \beta^+]$, which are superpositions of delta-functions with masses on the lattice

$$\Lambda_N = \left(-\frac{2^N}{2^{(\alpha/2)N}}, \frac{-2^N + 2}{2^{(\alpha/2)N}}, \ldots, \frac{2^N}{2^{(\alpha/2)N}}\right). \tag{6.7}$$

We take this lattice because it is the lattice over which the variable $v = 2^{-(\alpha/2)N} \sum_{i=1}^{2^N} \sigma_i$ takes its values when $\sigma_i = \pm 1$, however the theorem quoted below could be phrased for more general lattices.

Furthermore assume that there is a family π_β, $\beta \in [\beta^-, \beta^+]$, which verifies the hypotheses of Theorem 1 and which interpolates the masses of $\bar{\pi}_\beta$:

$$(\text{mass of } \bar{\pi}_\beta \text{ in } v) \propto \pi_\beta(v)\,, \tag{6.8}$$

then:

Theorem 2. Under the above assumptions the functions $\hat{K}^n \circ \bar{\pi}_\beta$ can be interpolated from functions with the same properties as those listed in Theorem 1 for the functions $\hat{K}^n \circ \pi_\beta$.

Corollary. There is a $\beta_c \in (\beta^-, \beta^+)$ such that for all t_1, $t_2 \in [-\infty, \infty]$ and $m = 1, 2, \ldots$

$$\lim_{L\to\infty} \int_{t_1}^{t_2} v^m \hat{K}^L \circ \bar{\pi}_{\beta_c}(v)\, dv = \int_{t_1}^{t_2} v^m \pi_G^{(\infty)}(v)\, dv\,. \tag{6.9}$$

The following theorem, stated but not proven in [4], describes what happens for $\beta \neq \beta_c$ but close enough to it.

Theorem 3. Under the same assumptions of Theorem 1, if $\beta < \beta_c$ and close enough to β_c, then there exists $\chi_l(\beta)$ such that

$$\lim_{L\to\infty} \int_{v_1}^{v_2} \pi^{(L)}_{L,\sqrt{2}}(v)\, dv \equiv \lim_{L\to\infty} \int_{v_1}^{v_2} \hat{K}^L \circ \pi_\beta\left(v(\sqrt{2}/2^{\alpha/2})^L\right) dv = \tag{6.10}$$

$$= (2\pi\chi_l(\beta))^{-\frac{1}{2}} \int_{v_1}^{v_2} \exp\left[-v^2/2\chi_l(\beta)\right] dv$$

for all v_1, $v_2 \in (-\infty, \infty)$ and

$$\chi_l(\beta) \underset{\beta\to\beta_c}{\simeq} (\beta_c - \beta)^{-1}\,. \tag{6.11}$$

The results (6.6) and (6.10) confirm the conjectures made in Sect. **2** about the long-order correlations.

7. – The case $\alpha > \frac{3}{2}$ and the ε-expansion.

In this Section we consider the case $\frac{3}{2} < \alpha < 2$ where a critical point still exists but the « Gaussian case » has no longer the desired structure since the linear approximation to $\hat{K}$ around this point possesses *two* or more eigenvalues larger than one. We want to give heuristic arguments that at the point $\alpha = \frac{3}{2}$ a new set of fixed points branches off from the Gaussian fixed point and that the linear approximation to $\hat{K}$ at these points seems to have on some natural domain (for $\alpha > \frac{3}{2}$) a single eigenvalue larger than one. This eigenvalue gives rise, as we will see, to a « nonclassical » behaviour.

In our argument we take α larger than $\frac{3}{2}$ only by a slight amount of order ε and expand the fixed-point equations in ε; for convenience we shall in fact define $\varepsilon = 2\zeta^4 - 1$. The procedure that we follow is a simple example of the so-called ε-expansion which has been used to calculate in the context of the renormalization group the deviations from the classical exponents for a dimensionality smaller than 4 $(\varepsilon = 4 - d$ in this case [11, 14]). The same method has been employed also to calculate, as in the present case, the deviation from the classical exponents for interactions falling off faster than $r^{-\frac{3}{2}}$ in ordinary Ising models [15, 16].

Let us consider a distribution $\pi_\varepsilon(x)$, analytic in x near $x = 0$ and of the form

$$\pi_\varepsilon(x) \simeq \exp[(-a_0 + r)x^2 - bx^4] + O(\varepsilon^3)\,, \tag{7.1}$$

and falling exponentially fast at infinity, which is a fixed point for the renormalization group. This equation determines r and b up to second order in ε, the value of a_0 is that corresponding to the Gaussian fixed point. We shall assume that r and b have asymptotic expansions in ε near $\varepsilon = 0$. A direct calculation yields then

$$\hat{K}\circ\pi_\varepsilon(x) \simeq \exp[(-a_0 + r')x^2 - b'x^4] + O(\varepsilon^3)\,, \tag{7.2}$$

where

$$r' = 2\zeta^2 r - \frac{6\zeta^2 b}{C + 2a_0}\left(1 + \frac{2r}{C + 2a_0} - \frac{6b}{(C + 2a_0)^2}\right), \tag{7.3}$$

$$b' = 2\zeta^4 b - \frac{36\zeta^4 b^2}{(C + 2a_0)^2}, \tag{7.4}$$

and $2\zeta^4 = 1 + \varepsilon$ (*i.e.* α slightly larger than $\frac{3}{2}$).

From these equations one can heuristically derive the first-order term of the asymptotic expansion of r and b; consider (7.3) and (7.4) as mappings of the plane into itself and see whether fixed-point solutions can be obtained. An obvious solution is of course $r^* = b^* = 0$, which is the Gaussian fixed point. We have, however, already noticed that this fixed point does not have the desired structure for the description of ordinary critical points in case $\alpha > \frac{3}{2}$. A new type of fixed point can also be found and is given by

$$\left|\begin{array}{l} r^*(\varepsilon) = \dfrac{C + 2a_0}{6(2 - \sqrt{2})}\,\varepsilon + \ldots \\[2ex] \text{and} \\[1ex] b^*(\varepsilon) = \dfrac{(C + 2a_0)^2}{18}\,\varepsilon + \ldots\,. \end{array}\right. \tag{7.5}$$

Consider now the linear approximant to $\hat{K}$ around this fixed point π^* in the space ∂K of distributions given by

$$\pi(x) = \pi^*(x)\big(1 + \varphi(x)\big)\,, \tag{7.6}$$

where $\varphi(x)$ is bounded at infinity and analytic near zero

$$\varphi(x) = \delta_1 x^2 + \delta_2 x^4 + \delta_3 x^6 + \dots . \tag{7.7}$$

To proceed one has to make the additional assumption that the space ∂K is invariant under $\hat{K}$. Then the action of $\hat{K}$ on π results in replacing δ_n by δ'_n for $n = 1, 2, \dots$; a straightforward calculation yields the following result for the *linearized* expressions for δ'_n:

$$\begin{cases} \delta'_1 = \left(\sqrt{2} + \frac{1}{6}\sqrt{2}\,\varepsilon\right)\delta_1 + \left(3\sqrt{2} + \left(1 + \frac{7}{6}\sqrt{2}\right)\varepsilon\right)\frac{\delta_2}{C + 2a_0} + \\ \qquad\qquad + \frac{15}{2}\sqrt{2}\left(1 + \frac{1}{2}\varepsilon\right)\frac{\delta_3}{(C + 2a_0)^2} + \dots + O(\varepsilon^2)\,, \\ \delta'_2 = (1 - \varepsilon)\,\delta_2 - 15(1 + \varepsilon)\frac{\delta_3}{C + 2a_0} + \dots + O(\varepsilon^2)\,, \\ \delta'_3 = 2^{-\frac{1}{2}}\left(1 + \frac{3}{2}\varepsilon\right)\delta_3 + \dots + O(\varepsilon^2)\,, \qquad \text{etc.} \end{cases} \tag{7.8}$$

The linear transformation defined by (7.8) has a triangular form, its eigenvalues are simply the diagonal elements. When ε is positive and small there is a single eigenvalue larger than one with the value

$$\lambda_1 = \sqrt{2} + \tfrac{1}{6}\sqrt{2}\,\varepsilon + O(\varepsilon^2)\,. \tag{7.9}$$

According to formula (5.28) this value of λ_1 corresponds to a value of γ given by

$$\gamma = \frac{(\alpha - 1)\ln 2}{\ln\left(\sqrt{2} + \frac{1}{6}\sqrt{2}\,\varepsilon\right)} + O(\varepsilon^2)\,. \tag{7.10}$$

The value of α can be expressed as $\alpha = \frac{3}{2} + \varepsilon/2\ln 2$, consequently one has

$$\gamma = 1 + \frac{2}{3}\frac{\varepsilon}{\ln 2} + O(\varepsilon^2)\,. \tag{7.11}$$

This value agrees with the value obtained in [16] for long-range Ising models. (Notice that the definition of ε in this reference differs by a factor $1/\ln 2$ from the present definition.)

The value of η remains classical as long as a fixed point with a finite second moment is reached.

We note that the above evaluations may be relevant only for models $\{H, \pi\}$ for which one can check that, for β close enough to β_c and for some L_0, $\pi_{L_0}^{(L_0)}$ is close to π^* in the sense that it can be expressed by (7.6) within the space ∂K and « small » (in the sense that all the coefficients δ should be small). Such a property is, of course, very difficult to check in any particular model: we have already seen what kind of results the analogous problem gives rise to in the case of the Gaussian fixed point (see Sect. **6**).

For some problems it might even turn out that the fixed point relevant to the critical behaviour is the above π^* but the deviations φ have to be allowed to vary in a different space: the spectrum might then be different. One may be helped in this search for the right space by *a priori* considerations based on properties which are conserved under the action of $\hat{\mathcal{R}}$ (see Sect. **9**).

8. – Surface terms.

In this Section we study the influence of models with additional interactions, which can be interpreted as surface terms, on the long-order and short-order correlations. We consider a more general hierarchical model defined by

$$H_L(D, B; \sigma_1, \dots, \sigma_{2^L}) = -D\sum_{p=1}^{L} 2^{-\alpha p}\sum_{m=1}^{2^{L-p}} M_{2m-1,p-1}M_{2m,p-1} - B\sum_{p=1}^{L}2^{-\alpha p}\sum_{m=1}^{2^{L-p}} M^2_{m,p}\,. \tag{8.1}$$

Up to now we only considered the case $B=0$, which was introduced by DYSON [2]. When we take $B=-\frac{1}{4}D$ the model is that studied by BAKER [3]. The value $D=0$ leads to another model studied by DYSON [1].

There is a close connection between the model (8.1) and the model that we studied so far. A direct calculation shows

$$H_L(D, B) \equiv H_L(\tilde{D}, 0) - \frac{B}{2^\alpha - 1}\sum_x \sigma_x^2 + \frac{B2^{-\alpha L}}{2^\alpha - 1} M^2_{1,L} \tag{8.2}$$

with

$$\tilde{D} = D + 2B/(1-2^{-\alpha})\,. \tag{8.3}$$

We restrict our attention to cases where $B > -\frac{1}{2}(1-2^{-\alpha})D$, so that $\tilde{D} > 0$.

Consider now the model defined by the couple $\{H_L(D, B); \pi_0\}$ and compare it with the model defined by the couple $\{H_L(\tilde{D}, 0); \tilde{\pi}_0\}$ with

$$\tilde{\pi}_0(v) \simeq \exp\left[\frac{B}{2^\alpha - 1}v^2\right]\pi_0(v)\,; \tag{8.4}$$

it turns out then that the difference between the two models is given by the last term of (8.2). The contribution to the energy of this term per spin is of the order $2^{(1-\alpha)L}$, and vanishes for $L\to\infty$, at least if π_0 has compact support.

In this case this term will therefore be interpreted as a surface term. The question that we ask is to what extent this surface term influences the long-order and short-order correlations and the critical properties.

A direct consequence of the relation (8.2) is that the renormalization group operator $K_{D,B}$ belonging to the models defined above can be expressed as

$$K_{D,B}\circ\pi(v) = \exp\left[\frac{-B}{2^\alpha - 1}\, v^2\right]\cdot K\circ\tilde{\pi}(v)\,, \tag{8.5}$$

where K is the integral operator used so far (with $D = \tilde{D}$) and $\tilde{\pi}$ is given by (8.4). Iteration of this formula yields

$$K^R_{D,B}\circ\pi(v) = \exp\left[\frac{-B}{2^\alpha - 1}\, v^2\right] K^R\circ\tilde{\pi}(v)\,. \tag{8.6}$$

So we immediately see that the long-order correlations do depend on the presence of surface terms. (We shall discuss this point in more detail later.) This dependence on the surface terms of the long-order correlations should contrast the expected surface term independence of the short-order correlations. This is, of course, harder to check; in the case the model is assumed to have a Gaussian fixed point it is possible to see heuristically that at least the short–long-order parameter is independent of the surface terms for $B > -\frac{1}{4}D$.

To study the short-order behaviour we consider again the quantity

$$2^{-\alpha R}\langle M^2_{1,R}\rangle^{D,B}_\infty = \lim_{L\to\infty}\langle v^2;\, K^R_{D,B}\circ\pi_0\rangle^{D,B}_L\,, \tag{8.7}$$

where we use the notation of Sect. **5**. Suppose that $K^R\circ\tilde{\pi}_0$ tends (for $R\to\infty$) to the Gaussian fixed point.

By (8.6) one has

$$\lim_{R\to\infty} K^R_{D,B}\circ\pi_0(v)\Big/\int K^R_{D,B}\circ\pi_0(v)\,dv \simeq \exp\left[-\frac{B}{2^\alpha - 1}\, v^2\right]\cdot\pi^\infty_G(v)\,. \tag{8.8}$$

Let us for the moment interchange limits and insert this limit in (8.7). With the same method as used in Sect. **5** and employing again (8.6) we find

$$\lim_{R\to\infty} 2^{-\alpha R}\langle M^2_{1,R}\rangle^{D,B}_\infty = \lim_{L\to\infty}\left\{-2^{-L}\frac{\partial}{\partial r}\left[\ln\int_{-\infty}^{\infty}\exp\left[-\frac{B}{2^\alpha - 2}\, v^2\right] K^L\circ\pi_r(v)\,dv\right]_{r=0}\right\},$$

where

$$\pi_r(v) = \exp\left[-(a + r)\, v^2\right]$$

and

$$a = \frac{\frac{1}{4}\tilde{D}}{2^{\alpha-1} - 1} = \frac{\frac{1}{4}D}{2^{\alpha-1} - 1} + \frac{\frac{1}{2}B}{(2^{\alpha-1} - 1)(1 - 2^{-\alpha})}\,. \tag{8.9}$$

A direct calculation yields

$$\lim_{R\to\infty} 2^{-\alpha R}\langle M_{1,R}^2\rangle_\infty^{D,B} = \lim_{R\to\infty}\sum_{l=0}^{R-1}\frac{\frac{1}{2}2^{(\alpha-2)l}}{2a+C} + \tag{8.10}$$

$$+\lim_{R\to\infty} 2^{(\alpha-2)R}\frac{\int v^2\exp[-v^2(\frac{1}{4}D+B)/(2^{\alpha-1}-1)]\,\mathrm{d}v}{\int\exp[-v^2(\frac{1}{4}D+B)/(2^{\alpha-1}-1)]\,\mathrm{d}v}.$$

The first term in (8.10) is identical to the result obtained in Sect. **6**, whereas the second term vanishes for $R\to\infty$. In this heuristic consideration therefore the surface term does not appear to have an influence on the short–long-order behaviour. Note however that the second term in (8.10) vanishes provided $B>-\frac{1}{4}D$. In the case $B<-\frac{1}{4}D$ the interchange of the limits is clearly a more delicate matter.

A possible way of dealing rigorously with this problem might be to try to write recursion formulae for the short-order correlation functions. These functions, however, do not obey simple recursion formulae as the long-order correlation functions do, and one has to introduce a more complicated scheme.

Consider a system of 2^L spins and Hamiltonian (8.1) and define

$$\begin{cases} \underline{\sigma}_L = (\sigma_1, \ldots, \sigma_{2^L}), \\ \underline{\sigma}_R = (\sigma_1, \ldots, \sigma_{2^R}), \\ \underline{\sigma}'_L = (0, 0, \ldots, \sigma_{2^R+1}, \ldots, \sigma_{2^L}). \end{cases} \tag{8.11}$$

Define further

$$\begin{cases} M'_{1,L} = \sum_{l=2^R+1}^{2^L}\sigma_l, \\ U_L \;= \tilde{D}\sum_{l=0}^{L-R}2^{-\alpha(R+l)}M_{2,R+l}, \end{cases} \tag{8.12}$$

and observe that

$$H_L(D,B;\underline{\sigma}) = H_R(D,B;\underline{\sigma}_R) + H_L(D,B;\underline{\sigma}'_L) - \tag{8.13}$$

$$-M_{1,R}U_L + \frac{B}{2^\alpha-1}\{2^{-\alpha L+1}M_{1,R}M'_{1,L} + (2^{-\alpha L}-2^{-\alpha R})M_{1,R}^2\}.$$

Therefore if

(8.14) $\quad W_L^{(R)}(U, M')\,\mathrm{d}U\,\mathrm{d}M' =$ {probability with respect to the Gibbs distribution of $\{H(D,B);\pi_0\}$, conditioned to $\sigma_1=\ldots=\sigma_{2^R}=0$, that the variables U_L and $M'_{1,L}\in(U, U+\mathrm{d}U)\times(M', M'+\mathrm{d}M')$}, we find

$$\pi_R^{(L)}(M) \simeq \pi_{R,1}^{(R)}(M) \exp\left[-\frac{B}{2^\alpha-1}(2^{-\alpha L}-2^{-\alpha R})M^2\right]\cdot \int W_R^{(L)}(U, M') \exp\left[MU-\frac{2B}{2^\alpha-1}(2^{-\alpha L})M'M\right] \mathrm{d}U\,\mathrm{d}M'. \tag{8.15}$$

If we introduce rescaled variables

$$\begin{cases} u = 2^{(\alpha/2)R}\, U = \tilde{D}\sum_{l=0}^{L-R} 2^{-(\alpha/2)l}(2^{-(\alpha/2)(R+l)}\, M_{2,R+l}), \\ v = 2^{-(\alpha/2)R}\, M, \\ v' = 2^{-(\alpha/2)L}\, M', \end{cases} \tag{8.16}$$

and define

$$\overline{W}_R^{(L)}(u, v') \simeq W_R^{(L)}(2^{-(\alpha/2)R}\, U, 2^{(\alpha/2)L}\, M'), \tag{8.17}$$

then we obtain

$$\pi_R^{(L)}(v) \simeq \pi_R^{(R)}(v) \exp\left[\frac{B}{2^\alpha-1}(1-2^{\alpha(R-L)})v^2\right]\cdot \int \overline{W}_R^{(L)}(u, v') \exp\left[uv'-\frac{2B}{2^\alpha-1}(2^{(\alpha/2)(R-L)})vv'\right] \mathrm{d}u\,\mathrm{d}v'. \tag{8.18}$$

When we employ the relation (8.6) we obtain

$$\pi_R^{(L)}(v) \simeq \exp\left[\frac{B2^{\alpha(R-L)}}{2^\alpha-1}v^2\right] K^R\circ\tilde{\pi}_0(v)\cdot \int \overline{W}_R^{(L)}(u, v') \exp\left[uv'-\frac{2B2^{(\alpha/2)(R-L)}}{2^\alpha-1}vv'\right] \mathrm{d}u\,\mathrm{d}v'. \tag{8.19}$$

The factor in front of the integral, in the limit $L\to\infty$, no longer depends on the surface terms. One would like to obtain enough information about the probability $\overline{W}$ to see whether the second factor shares this property. This is not a hopeless problem since $\overline{W}_R^{(L)}$ satisfies a recursion relation that we want to derive now.

Consider a system of size 2^{L+1} and observe that

$$H_{L+1}(D, B; \underline{\sigma}'_{L+1}) = H_L(D, B; \underline{\sigma}''_{L+1}) + H_L(D, B; \underline{\sigma}'_L) - - D2^{-\alpha L} M'_{1,L} M_{2,L} - B2^{-\alpha L}(M_{2,L}+M'_{1,L})^2, \tag{8.20}$$

where $\underline{\sigma}''_{L+1} = \{\sigma_{2^L+1}, \dots, \sigma_{2^{L+1}}\}$. Hence it follows that

$$(8.21)\qquad \overline{W}_R^{(L+1)}(u, v') = \exp[(B2^\alpha + D\zeta^2)v'^2]\cdot \cdot\int \overline{W}_R^{(L)}\left(u - \frac{\tilde{D}(\zeta v' + y)}{2^{(\alpha/2)(L-R)}}, \zeta v' - y\right)\exp[-Dy^2]\pi_L^{(L)}(\zeta v' + y)\,dy\,.$$

The above recursion formulae seem quite natural, but so far it has not been possible to really employ them and the investigation of the short-order correlations remains an open problem.

We now come back to a discussion of the long-range correlations. Consider again eq. (8.6) for $1 < \alpha < \frac{3}{2}$ and suppose that $\hat{K}^R\circ\tilde{\pi}$ tends to the Gaussian fixed point. Let us proceed heuristically and assume that for some R_0 one arrives at a form

$$(8.22)\qquad \hat{K}^{R_0}\circ\tilde{\pi} = \frac{1}{\text{normalization}}\exp[-av^2 + rv^2 - bv^4] + O(r^2, b^3)\,,$$

where a has the value $a = \frac{1}{4}\tilde{D}/(2^\alpha - 1)$ corresponding to the Gaussian fixed point and r and b are both small. By the same method as used in the ε-expansion one obtains for $\hat{K}^{R_0+l}\circ\tilde{\pi}$ a similar form with parameters r_l and b_l given by

$$b_l = (2\zeta^4)^l\beta_l\,, \qquad \beta_l \xrightarrow[l\to\infty]{} \beta_\infty > 0\,,$$

and

$$(8.23)\qquad r_l = (2\zeta^2)^l[r - \varrho_l]\,, \qquad \varrho_l \xrightarrow[l\to\infty]{} \varrho_\infty + \varrho'\zeta^{2l}\,.$$

In this approximation the system is critical when we set $r = \varrho_\infty$ since then both b_l and r_l tend to zero. From (8.6) we conclude that

$$(8.24)\qquad \hat{K}_{D,B}^{R_0+l}\circ\pi_0(v) = \frac{1}{\text{normalization}}\exp\left[-\frac{B+\frac{1}{4}D}{2^{\alpha-1}-1}v^2 + (2\zeta^4)^l(\varrho'v^2 - \beta_\infty v^4)\right].$$

In case $B > -\frac{1}{4}D$ one obtains in the limit $l\to\infty$ a limiting function with a finite width. This implies that in this case the long-long range the critical exponent η_l is not affected by the surface terms, the corresponding amplitudes however do change with the surface terms. The conclusions that we obtain here in a heuristic way are made rigorous in [5], BLEHER and SINAII prove indeed the convergence, under the repeated action of $\hat{K}_{D,B}$, $B > -\frac{1}{4}D$, of initial distributions satisfying the conditions of Theorem 1, towards the Gaussian fixed point. Their bounds on the sequence of distribution functions are quite similar to those assumed in (8.24).

The next case to consider is $B = -\frac{1}{4}D$. In order to obtain a limiting function with finite width one has to rescale the variable v to

$$(8.25)\qquad v' = (2^{\frac{1}{4}}\zeta)^l v = (2^{-\frac{3}{4}})^l M\,.$$

If one neglects the fact that (8.24) is, in fact, an approximation valid for small v and takes it literally also for large v, then one finds in this scale

$$\pi^{(R_0+l)}_{R_0+l2,^{3/4}}(v') = \frac{1}{\text{normalization}} \exp\left[-\beta_l v^4 + (2\zeta^4)^{l/2}\varrho' v^2\right], \tag{8.26}$$

which for $l \to \infty$ seems to tend to a distribution with a finite width L. The change of scale implies that the exponent η has now a different value, namely $\eta = \frac{3}{2}$. This result was obtained along similar lines in [17].

In this (and the next) case a rigorous proof is still lacking. The results of BLEHER and SINAII [18] provide, however, for v large enough the estimate

$$0 \leqslant \pi^{(R_0+l)}_{R_0+l}(v)/\pi^{(\infty)}_{G}(v) \leqslant \exp\left[-d(2\zeta^4)^l v^4\right]. \tag{8.27}$$

The only rigorous conclusion therefore is that the rescaling factor chosen in (8.25) is at least a lower bound for the true scaling factor.

The last case is $B < -\frac{1}{4}D$; again a different scale has to be used, namely

$$v' = (2^{\frac{1}{2}}\zeta^2)^l v = 2^{(\alpha/2-3/2)l} M. \tag{8.28}$$

One obtains then, again heuristically

$$\pi^{(R_0+l)}_{R_0+l,2^{3/2-\alpha/2}}(v') = \frac{1}{\text{normalization}} \exp\left[(2\zeta^4)^{-l}(A_l v^2 - \beta_l v^4)\right], \tag{8.29}$$

where

$$A_l = \frac{\frac{1}{4}D - B}{2^{\alpha-1}-1} - \varrho'(2\zeta^4)^l.$$

In the limit $l \to \infty$ one obtains a distribution of finite width ($\neq 0$) consisting of two delta-function located symmetrically with respect to and at a finite distance from the origin. The new value of η which is implied in this case is $\eta = \alpha$.

9. – Conservation laws.

We conclude this paper with a few rigorous remarks which might be relevant in the study of the renormalization group equation for the hierarchical model. As we have seen in Sect. 7 information that can *a priori* exclude certain ansatz for the fixed-point $\pi^{(\infty)}$ and linear space ∂K can be very useful since they may give some information about the spectrum of $\hat{T}$. It is in this context that we point out the following « conservation laws » which are most easily formulated in terms of the renormalization group operator $\hat{K}_{D,-\frac{1}{4}D}$ introduced by BAKER.

From the definition (8.5) one easily finds the explicit form

$$\hat{K}_{D,-\frac{1}{4}D}\circ\pi(v) = \frac{1}{\text{normalization}}\int \pi(\zeta v + y)\pi(\zeta v - y)\exp[-D2^{-\alpha}y^2]\,dy\,. \tag{9.1}$$

Let us assume that $K^L_{D,-\frac{1}{4}D}\circ\pi_0$ tends weakly to a fixed point $\pi^{(\infty)}$, *i.e.*

$$\lim_{L\to\infty}\int_{v_1}^{v_2}\hat{K}^L_{D,-\frac{1}{4}D}\circ\pi_0(v)\,dv = \int_{v_1}^{v_2}\pi^{(\infty)}(v)\,dv \tag{9.2}$$

for all v_1, v_2. We can then formulate the following conservation laws:

i) Conservation of the « bells »: if $\pi_0(v)$ is a piecewise smooth even function which is monotonic for $v\geqslant 0$ (*i.e.* if π_0 is « bell shaped »), then this property is shared by $\hat{K}^L_{D,-\frac{1}{4}D}\circ\pi_0$. The limit $\pi^{(\infty)}$ is then also bell shaped (but no longer necessarily piecewise smooth).

ii) Conservation of the Fourier type; if π_0 is a positive normalized measure with positive Fourier transform, then $\hat{K}^L_{D,-\frac{1}{4}D}\circ\pi_0$ is of the same type. If $\pi^{(\infty)}$ is also normalizable, then $\pi^{(\infty)}$ is also of positive Fourier type.

iii) Conservation of the Griffiths' type; we recall from Sect. **3** that π_0 is said to be of Griffiths' type if either

$$\pi_0(v) \simeq \sum_{t_1,\dots,t_N}^{\pm 1}\exp\left[\sum_{i,j}^{1,N} J_{i,j}\,t_i t_j\right]\delta\left(N^{-\alpha}\sum_{i=1}^{N} t_i - v\right) \tag{9.3}$$

with $J_{i,j}\geqslant 0$, or π_0 is a weak limit of distributions π_n satisfying (9.3) such that there exist a $C > 0$ and $\varepsilon > 0$ with

$$\pi_n(v) \leqslant \exp[-\varepsilon v^2]\,. \tag{9.4}$$

The condition (9.4) is an adaptation of (3.6) to Baker's formulation of the hierarchical model. The property of being of Griffiths' type is conserved under the action of $\hat{K}_{D,-\frac{1}{4}D}$ provided D is such that for all i, j

$$0 < D2^{-\alpha}N^{-2\alpha} \leqslant 4J_{i,j}\left(1 - \frac{1}{4\zeta^2}\right); \tag{9.5}$$

in the case that π_0 is obtained from a sequence this relation should hold term by term. Notice that the condition (9.5) is empty in the case of a π_0 of the Ising type. The fixed point $\pi^{(\infty)}$ is also of Griffiths' type provided that $\hat{K}^L_{D,-\frac{1}{4}D}\circ\pi_0$ satisfies the bound (9.4).

One could have started the discussion of the conservation laws also in the context of another version of the hierarchical model, that of DYSON [1], leading to the renormalization operator $\hat{K}_{0,B}$. In that case Griffiths' type is conserved without an extra condition of the type (9.5).

The conservation laws i) and ii) are then no longer valid but hold modulo a Gaussian factor.

A proof of the above statements can be found in [19].

The fixed point which arises in the ε-expansion (for $\alpha > \frac{3}{2}$) is neither bell shaped nor of positive Fourier type; we can then use the conservation laws i) and ii) to conclude that the above nontrivial fixed point discussed in Sect. **7** cannot become relevant for a critical point of a model with a π_0 which is either bell shaped or of positive Fourier type. This leads to the conjecture that models of this type might be associated with the Gaussian fixed point even for $\alpha \geqslant \frac{3}{2}$ (*).

Of course also other, nontrivial, fixed points might become relevant in these cases.

The nontrivial fixed point found in the ε-expansion is compatible with a Griffiths' type $\pi^{(0)}$, hence it is possible to encounter it in the theory of the Ising-hierarchical model.

(*) The domain of $\hat{T}$ should exclude the eigenfunction H_4; in such a case the critical point will remain classical even for $\alpha \geqslant \frac{3}{2}$. The Gaussian model [20] is an example.

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