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The Lavoisier Law [1] and the Critical Point.

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Summary. — This paper contains mainly an exposition of Wilson's theory of the Kadanoff renormalization group.

1. – Introduction.

BLEHER and SINAI in a recent work on the hierarchical models [2] shed some light on the feasibility of a rigorous approach to the normalization group theory of critical phenomena. The aim of this paper is to discuss, in the frame of a quite general model, the formulation of this theory in order to try to single out a set of assumptions on which the general validity of this program should rely.

The paper is divided in two parts: in the first we discuss for a general model the basic ideas of Wilson's theory of the Kadanoff renormalization group and the associated scaling laws. We put some effort in trying to isolate four assumptions which, if suitably formulated, should be checked in models.

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(1) « We must lay it down as an incontestable axiom, that in all operations of art and nature, nothing is created: an equal quantity of matter exists both before and after the experiment ... and nothing takes place beyond changes and modifications of these elements ». A. L. LAVOISIER: *Traité èlèmentaire de chimie présenté dans un ordre nouveau* (Paris, 1789) (new edition, Paris, 1937).

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(2) P. H. BLEHER and JA. G. SINAI: *Comm. Math. Phys.*, **33**, 23 (1973).

In the second part we discuss the ε-expansion and the related calculations of the critical exponents in the case of the hierarchical model where the approximations can be controlled.

We shall only deal with Ising models on a d-dimensional square lattice Z^d at zero external field and above the critical temperature. The treatment for the general case (h, T arbitrary) can be followed along the same lines with some minor changes and it is left out mainly for the sake of a simpler formalism.

As the reader will soon realize we are inspired by the very clear exposition of WEGNER (3) and by the rigorous results of BLEHER and SINAI (2) and, in some sense, we are only putting together the qualitative aspects of the two papers without being able to be as general as the first or as rigorous as the second, but we suggest a number of conjectures, still unproven even in simple models, which may help the mathematical physicist to the understanding of the mathemetical aspects of the problem.

2. – The Ising model.

We shall call here Ising model a somewhat more extended class of models.

On each site $\xi \in Z^d$ sits a spin $\sigma_\xi \in (-\infty, +\infty)$. The energy of a configuration contained in a box Λ is given by

$$H^{(\Lambda)}(\underline{\sigma}) = -\sum_k \sum_{\xi_1, \ldots, \xi_k \in \Lambda} f^{(k)}_{\xi_1, \ldots, \xi_k}(\sigma_{\xi_1}, \ldots, \sigma_{\xi_k}), \tag{2.1}$$

where the functions $f^{(k)}_{\xi_1, \ldots, \xi_k}(\sigma_{\xi_1}, \ldots, \sigma_{\xi_k})$ are defined for all k's and $\xi_1, \ldots, \xi_k \in Z^d$ and are even in the σ's and periodic over the simultaneous lattice translations of all the labels $\xi_1, \ldots, \xi_k$ (with a period which may depend on k). The functions $f^{(k)}$ are assumed to vanish if one of their arguments vanishes. For convenience we shall restrict the side L of the box Λ to be a power of 2, $L = 2^N$, and the periods of the potentials will also be restricted to be powers of 2.

If $L = 2^N$ we shall write $H^{(N)}(\underline{\sigma})$ instead of $H^{(\Lambda)}(\underline{\sigma})$.

A model (H, π) is characterized by an H of the above type and by what we shall call the «free-spin probability measure» $(\pi(\sigma)\,\mathrm{d}\sigma)$ which would be the Gibbs distribution of each spin if H were zero. The measure $\pi(\sigma)\,\mathrm{d}\sigma$ must be even and normalized so that $\int \pi(\sigma)\,\mathrm{d}\sigma = 1$.

The partition function for the model (H, π) will be, if Λ is a box with side 2^N,

$$Z_N(H, \pi) = \int \exp[-H^{(N)}(\underline{\sigma})] \prod_{\xi \in \Lambda} \pi(\sigma_\xi)\,\mathrm{d}\sigma_\xi . \tag{2.2}$$

(3) F. J. WEGNER: *Phys. Rev. B*, **5**, 4529 (1972).

The above generality is necessary for the coming discussion even though we may eventually be interested only in the nearest-neighbour spin-$\frac{1}{2}$ Ising model which is given by $f^{(k)}_{\xi_1, \dots, \xi_k} = 0$ unless $k = 2$ and $|\xi_1 - \xi_2| = 1$

$$\pi(\sigma) = \frac{\delta(\sigma + 1) + \delta(\sigma - 1)}{2}, \qquad f^{(2)}_{\xi_1,\xi_2}(\sigma_{\xi_1}, \sigma_{\xi_2}) = J\sigma_{\xi_1}\sigma_{\xi_2}. \tag{2.3}$$

Formulae (2.1), (2.2) and the example (2.3) should clarify the meaning of the above somewhat abstract setting. We shall assume that the functions $f^{(k)}(\sigma_{\xi_1}, \dots, \sigma_{\xi_k})$ and $\pi(\sigma)$ have enough « good » properties (like not too long range, stability, etc. ...) so that the thermodynamic limit for the free energy and cor relation functions of the model (H, π) exist and are boundary condition inde pendent.

In other words, we assume that the model (H, π) defines a unique Gibbs random field.

It is this last assumption that will force us to work only for $T \geqslant T_c$.

3. – Block spins. Renormalization group.

The block spins were introduced, in the context of the theory of the critical point, by Kadanoff ([4]). This concept also arises naturally in probability theory (and precisely in the limit theorems) and the connection between the limit theorems of probability theory and the theory of the critical point is probably rather deep ([2,5,6]). Let $X = (x_1, \dots, x_d)$ be d integers and think of Z^d as the union of adjacent boxes (say squares) with side 2^n; we denote by $\{x, n\}$ the box

$$\{\xi \in Z^d, 2^n x_i \leqslant \xi_i < 2^n(x_i + 1), i = 1, \dots, d\}, \tag{3.1}$$

where ξ_i are the co-ordinates of ξ (integers).

Define

$$\nu^{(n)}_{x,\varrho} = \frac{\sum_{\xi \in \{x,n\}} \sigma_\xi}{2^{nd\varrho/2}}, \tag{3.2}$$

where $0 < \varrho \leqslant 2$ is an arbitrary « scaling » parameter. When the value of ϱ or n or x is deducible from the context we shall simply write $\nu_{x,\varrho}$ or ν_x or $\nu^{(n)}$ for $\nu^{(n)}_{x,\varrho}$. Consider a model (H, π) and the associated (unique) infinite-volume Gibbs equilibrium state. Then the σ_ξ and the $\nu^{(1)}_{x,\varrho}$ are random variables with well-defined probability distributions in the equilibrium state of (H, π): they will be the « σ-field » and the « $\nu^{(1)}$-field » of (H, π).

([4]) L. P. Kadanoff: *Physics*, **2**, 263 (1966).

([5]) G. Jona Lasinio: *The renormalization groups: a probabilistic view*, Padova preprint (1974).

([6]) G. Gallavotti, H. J. F. Knops and G. C. Martin Löf: preprint (1974).

We ask the question: does there exist a model (H', π') whose σ-field is distributed as the $\nu^{(1)}$-field of (H, π)? It is easy to give a formal answer to the above question. Let us first define

$$\pi'(\nu) = \frac{\int \prod_{\xi\in\{x,1\}} \left(\pi(\sigma_\xi)\,\mathrm{d}\sigma_\xi\right) \exp[-H^{(1)}(\underline{\sigma})]\, \delta\left(\frac{\sum_{\xi\in\{x,1\}} \sigma_\xi}{2^{d\varrho/2}} - \nu\right)}{Z_1(H, \pi)}, \tag{3.3}$$

where $\{x, 1\}$ is a unit cell (containing 2^d points) and $H^{(1)}$ is the Hamiltonian H restricted to the box $\{x, 1\}$. In other words π' is the distribution that the block spins $\nu^{(1)}_{x,\varrho}$ would have if all the interactions between spins in different blocks were turned off. Then the partition function $Z_N(H, \pi)$ can be written as

$$Z_N(H, \pi) = \int \prod_x \pi'(\nu_x)\,\mathrm{d}\nu_x Z_N(H, \pi, \underline{\nu}), \tag{3.4}$$

where the product over x runs over the blocks of 2^d spins into which the box Λ with side 2^N is divided and $Z_N(H, \pi, \underline{\nu})$ is the partition function over the volume Λ associated to the σ-spin distribution

$$P(\underline{\sigma})\,\mathrm{d}\underline{\sigma} \propto \prod_x \delta\left(\frac{\sum_{\xi\in\{x,1\}} \sigma_\xi}{2^{d\varrho/2}} - \nu_x\right) \prod_{\xi\in\{x,1\}} \pi(\sigma_\xi)\,\mathrm{d}\sigma_\xi \exp[-H(\underline{\sigma})], \tag{3.5}$$

i.e. to the random field generated by the Hamiltonian (H, π) with a constraint: the allowed configurations are those which precisely produce the set of block spins which appears in the label. We can now define the effective Hamiltonian as

$$\tilde{H}'(\underline{\nu}) = -\ln \frac{Z_N(H, \pi, \underline{\nu})}{Z_N(H, \pi, \underline{0})}, \tag{3.6}$$

the $\sim$ indicating that $\tilde{H}'$ depends on N.

The above $\tilde{H}'(\underline{\nu})$ can be written as

$$\tilde{H}'(\underline{\nu}) = -\sum_k \sum_{\xi_1, \ldots, \xi_k} \tilde{f}'^{(k)}(\nu_{\xi_1}, \ldots, \nu_{\xi_k}) \tag{3.7}$$

with the functions $\tilde{f}'$ vanishing if one of their arguments vanishes. For instance

$$\tilde{f}'^{(1)}(\nu_x) = \ln \frac{Z_N(H, \pi, \hat{\underline{0}}, \nu_x)}{Z_N(H, \pi, \underline{0})}, \tag{3.8}$$

where $\hat{\underline{0}}, \nu_x$ is the set of block spins which are 0 for all boxes except the one with label x. Similar formulae can be recursively found for the other functions. We observe that the ratio of partition functions in (3.8) can be interpreted as the average of a local quantity in the Gibbs random field, generated by the Hamiltonian (H, π) (restricted to the box Λ), with a constraint on the allowed configurations $\underline{\sigma}$: the allowed configurations are those which produce sets of block spins with zero magnetization. If we *assume* that also this modified (H, π)-model has a unique equilibrium state, then we should expect that the following limits exist:

$$f'^{(k)}(\nu_{\xi_1}, \ldots, \nu_{\xi_k}) = \lim_{N \to \infty} \tilde{f}'^{(k)}(\nu_{\xi_1}, \ldots, \nu_{\xi_k}) . \tag{3.9}$$

We shall denote the constrained (H, π)-model by $(H, \pi)_{\underline{0}}$.

The limiting values $f^{(k)}$ together with π' define a new model (H', π') and the transformation from (H, π) to (H', π') will be denoted by

$$(H', \pi') = K_\varrho(H, \pi) . \tag{3.10}$$

We assume that also (H', π') is a « good model », *i.e.* it has a unique Gibbs random field which is precisely the same as the $\nu^{(1)}_{x,\varrho}$-field.

The conditions on (H, π) under which the above assumptions hold

1) $(H, \pi)_{\underline{0}}$ has a unique Gibbs random field,

2) (H', π') has a unique Gibbs random field which is precisely the $\nu^{(1)}_{x,\varrho}$-field,

are not known.

These two statements are not unreasonable and may be true for very general (H, π) which have a unique Gibbs distribution.

In the following we are going to assume 1) and 2) for all (H, π) of interest and, also, for the associated models of the form $K^n_\varrho(H, \pi)$, $n = 1, 2, \ldots$. We shall also assume that the interchange of the limits involved in assumption 2) allows one to deduce from the obvious relation

$$Z_N(H, \pi) = Z_N(H, \pi, \underline{0})\, Z_{N-1}(\tilde{H}', \pi') \tag{3.11}$$

the nontrivial one

$$f(H, \pi) = \Gamma(H, \pi) + \frac{1}{2^d} f(H', \pi') , \tag{3.12}$$

where

$$f(H, \pi) = \lim_{N \to \infty} \frac{1}{2^{Nd}} \ln Z_N(H, \pi) \tag{3.13}$$

and

$$\Gamma(H, \pi) = \lim_{N \to \infty} \frac{1}{2^{dN}} \ln Z_N(H, \pi, \underline{0}) = f\big((H, \pi)_{\underline{0}}\big) . \tag{3.14}$$

Hence, by iteration, if $(H_l, \pi_l) = K_\varrho^l(H, \pi)$,

$$f(H, \pi) = \sum_{k=1}^{l} \frac{f\big((H_{k-1}, \pi_{k-1})_\varrho\big)}{2^{d(k-1)}} + 2^{-dl} f(H_l, \pi_l) = \mu_l + 2^{-dl} f(H_l, \pi_l) , \tag{3.15}$$

where μ_l is defined here.

4. – The basic assumption for the scaling laws.

In this Section we list and briefly discuss the basic assumptions [4,7] that are needed for the derivation of the scaling laws in zero field. These assumptions seem quite natural and, therefore, imply some understanding of the mechanism behind the scaling laws.

I) The transformation K_ϱ in (3.11), which depends on ϱ only, is smooth in the neighbourhood of any (H, π) to which a unique Gibbs state is associated.

This assumption is rather loosely stated here and needs some comments. First the neighbourhood of (H, π) that appears in I) must consist of models $(\hat{H}, \hat{\pi})$ with $(\hat{H}, \hat{\pi})$ « close » to (H, π) but still such that the Gibbs equilibrium distribution for $(\hat{H}, \hat{\pi})$ is unique [8].

The second comment is a more explicit definition of what is meant by smoothness; we regard (H, π) as a sequence of « coupling functions » $(\pi, \{f^{(k)}_{\xi_1, \dots, \xi_k}\})$ and denote by $(H, \pi) + z(\delta H, \delta\pi)$ the model corresponding to the sequence $(\pi + z\,\delta\pi, \{f^{(k)}_{\xi_1, \dots, \xi_k} + z\,\delta f^{(k)}_{\xi_1, \dots, \xi_k}\})$. Then we wish that the model

$$K_\varrho\big((H, \pi) + z(\delta H, \delta\pi)\big) = K_\varrho(H, \pi) + z^w(\delta H', \delta\pi')$$

where w is a suitable nonnegative constant (*i.e.* independent of $(\delta H, \delta\pi)$) and $(\delta H', \delta\pi')$ is an increment of « order 1 » when $z \to 0$. This requirement should hold for increments $(\delta H, \delta\pi)$ which are not « too wild ». We do not specify what we really mean by « order 1 » and prefer to leave it to the reader to figure out the several possible interpretations of this sentence, which would allow the formal manipulations performed below.

In conclusion we can say that « smooth » has to be interpreted in the sense that K_ϱ satisfies a Hölder condition with constant w which is « direction »-independent at least for the interesting directions. One would be tempted to say

(7) K. G. WILSON: *Phys. Rev. B*, **4**, 3174, 3184 (1971).

(8) Hence if we consider the Ising model $(\beta_c \bar{H}_0, \bar{\pi}_0)$ at the critical temperature, the models $(\beta \bar{H}_0, \bar{\pi}_0)$ will be in some « neighbourhood » only if $\beta \leqslant \beta_c$.

that this statement should follow from the fact that the new coupling functions can be expressed as thermodynamic local averages in the constrained model $(H, \pi)_0$ (cf. (3.9)) which should not have long-range correlations simultaneously with (H, π) (because they are just different models) and therefore the local averages should depend smoothly on the parameters of $(H, \pi)_0$, *i.e.* of (H, π). Actually one would even say that $w = 1$.

This is, however, a very dangerous statement [9] and it is not unlikely that there are some interesting instances in which (H, π) and $(H, \pi)_0$ are simultaneously critical. Even in this case, however, nothing is against the Hölder continuity of the local thermodynamic averages as functions of the variation of (H, π) and the only questionable thing is the direction independence of w (we remember, however, that we are here discussing only the $h = 0$ case and so directions which lead to uneven interactions and π's are not allowed).

II) Consider the models (H, π) which give rise to a unique equilibrium state and such that there is a value ϱ for which

$$0 < \lim_{n\to\infty} \langle (\nu_{x,\varrho}^{(n)})^2 \rangle_{(H,\pi)} = \Delta < \infty . \tag{4.1}$$

Then the limit

$$\lim_{n\to\infty} K_\varrho^n(H, \pi) = (H^*, \pi^*) \tag{4.2}$$

exists and is nontrivial (*i.e.* we exclude $\pi^*(\nu) = \delta(\nu)$).

One should specify the sense of (4.2) and there are several possible alternatives. The simplest would be to interpret (4.2) as saying that the Gibbs random field of $K_\varrho^n(H, \pi)$ has a weak limit as $n \to \infty$ which is, also, a random field. The precise meaning of this statement is the following: let $P_\Lambda^{(n)}(\sigma_1, \ldots, \sigma_\Lambda) \cdot \mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda$ be the probability distribution, in the Gibbs random field of $K_\varrho^n(H, \pi)$, of the σ-spins of the box Λ, then for all test functions with compact support

$$\lim_{n\to\infty} \int P_\Lambda^{(n)}(\sigma_1, \ldots, \sigma_\Lambda)\varphi(\sigma_1, \ldots, \sigma_\Lambda)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda = \int P_\Lambda^{(\infty)}(\ldots)\varphi(\sigma_1, \ldots, \sigma_\Lambda)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda , \tag{4.3}$$

where $P_\Lambda^{(\infty)}(\sigma_1, \ldots, \sigma_\Lambda)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda$ is a probability measure which satisfies the

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[9] There is a very nice counterexample due to KASTELEYN to the general validity of such a statement: consider the two-dimensional Ising model with the constraint that the block spins $\{x, 1\}$ are all zero and furthermore the block configurations $\begin{smallmatrix}+&-\\-&+\end{smallmatrix}$ and $\begin{smallmatrix}-&+\\+&-\end{smallmatrix}$ are forbidden; this model is exactly solvable and has the same critical temperature as the original unconstrained Ising model! (KASTELEYN: private communication.)

necessary and obvious compatibility requirements which allow one to think of it as a reduced probability of a random field [10].

The existence of (4.2) in this weak sense is not a « too strong » requirement: in fact (4.1) means (if assumptions 1) and 2) of Sect. **3** are accepted)

$$\int P_\Lambda^{(n)}(\sigma_1, \ldots, \sigma_\Lambda)\sigma_1^2 \mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda \xrightarrow[n\to\infty]{} \Delta < \infty, \tag{4.4}$$

which allows, via simple compactness arguments, possibly by passing to a subsequence, to deduce that the $P_\Lambda^{(\infty)}(\sigma_1, \ldots, \sigma_\Lambda)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\Lambda$ exist and satisfy the compatibility requirements.

So the real assumption, if one interprets (4.2) in the above minimal sense, is that the limit exists over the whole sequence and not only on subsequences. Not everything of what follows would crumble if one only assumes (4.2) for subsequences: one would have to introduce the set of accumulation points of the sequence $K_\varrho^n(H, \pi)$ and use them instead of the simple point (H^*, π^*) (limit cycles). We call the above compactness arguments the Lavoisier law [1]; we leave to the reader to expound why (hint: (4.4) says that the « mass » of the probability distributions $P_\Lambda^{(n)}(\ldots)$ is forced to stay in a finite region while n varies) [11].

One may wonder why the exponent 2 in (4.1) plays such a special role. Actually one hopes that at least for short-range models with a π with compact support the same ϱ defined by (4.1) guarantees that

$$0 < \lim_{n\to\infty} \langle (\nu_{x,\varrho}^{(n)})^r \rangle_{(H,\pi)} = \Delta_r < \infty$$

for all choices of r.

Mathematically however this is not a consequence of (4.1) and therefore the special role of the exponent 2 is an essential part of assumption II). But again, as in the above discussed case of the « limit cycles », the theories that follows seem, at least partially, adaptable to more pathological situations.

Unfortunately such a minimal interpretation of (4.2) is not enough for our purposes: we shall also have to interpret (4.2) as saying that the limit random field, discussed above, can be generated as the unique equilibrium random field of a model (H^*, π^*) and the coupling functions of $K_\varrho^n(H, \pi)$ approach the coupling functions of (H^*, π^*) so that the formal manipulations which we are going to make on this basis are allowed.

Again II) is a rather well-posed mathematical problem and here it does

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(10) I. I. GHICHMAN and A. V. SCOROCHOD: *Theory of Random Processes* (Moscow, 1971); P. L. DOBRUSHIN: *Theory of prob. and applications*, **13**, 197 (1968); **25**, 458 (1970).

(11) The existence of the above limit in a sense stronger than the weak one just described can be rigorously established for noncritical cases cf. ref. (6).

not make much sense to specify it more by explicitly stating the convergence conditions [12].

Let us now consider a model of the form $(\beta H_0, \pi_0)$ with β a real parameter $\beta \geqslant \beta_c$.

Assume that β_c is the inverse critical temperature for the model with Hamiltonian H_0 and free-spin distribution π_0. By this we mean that the Gibbs equilibrium state of $(\beta H_0, \pi_0)$ is unique for $\beta \leqslant \beta_c$ and not unique for $\beta > \beta_c$.

Then by I), II) we can say that

$$K_\varrho^n(\beta H_0, \pi_0) = K_\varrho^n(\beta_c H_0, \pi_0) + (\beta - \beta_c)^w \Delta_n \tag{4.5}$$

with Δ_n of « order 1 » for n fixed and

$$K_\varrho^n(\beta_c H_0, \pi_0) = (H^*, \pi^*) + \varepsilon_n$$

with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ (in some sense to be specified).

III) The models $K_\varrho^n(\beta H_0, \pi_0)$ and $K_\varrho^n(\beta_c H_0, \pi_0)$ will start being different as soon as 2^n exceeds the correlation length, which, therefore, is determined by the condition that $(\beta - \beta_c)^w \Delta_n$ be of « order 1 » at β fixed. This is a rather strict interpretation of Kadanoff's idea that, close to the critical point, the distribution of the spins inside a box with side less than the correlation length is essentially equal to the one they would have if they were at the critical point [4].

IV) The difference Δ_n can be thought, if $\beta \sim \beta_c$, as belonging to a linear space $\partial(H^*, \pi^*)$ and the operator K_ϱ linearized around (H^*, π^*), along $\delta(H^*, \pi^*)$, has a spectrum with only one eigenvalue $\lambda > 1$.

Since it is well known that the spectrum of an operator depends on the space on which it acts, we have to specify that $\partial(H^*, \pi^*)$ must be the « minimal » space with the above property, *i.e.* the minimal space containing the relevant increments Δ_n. In general, by taking a space $\partial(H^*, \pi^*)$ too large, one may cause the linearized operator to have a very bad spectrum with as many points as wanted.

So this assumption seems quite unconstructive because its verification might mean a good knowledge of the behaviour of the model near the critical

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([12]) The existence of a ϱ such that (4.1) holds is essentially equivalent to the requirement that the pair correlation function in (H, π) behaves as a pure power law for large distances: if $\langle \sigma_0 \sigma_R \rangle \simeq R^{-(d-2+\eta)}$, then $\varrho = 1 + (2 - \eta)/d$. If this is not true, *i.e.* there are, say, logarithmic corrections, then one has to choose ϱ as a function of n but, usually, $\lim_{n\to\infty} \varrho_n$ will exist and one could try to give arguments similar to the ones used in the case of constant ϱ. We do not enter here in this discussion.

point. It seems that one could hope to apply it to some explicit case where there are reasons to put severe limitations on the space (H^*, π^*) (for instance in the simplest approximation scheme, which is the lowest-order ε-expansion ([13]), one guesses that $\partial(H^*, \pi^*)$ is a certain 2-dimensional space (see end of Sect. 7 for a more precise comment)).

It is therefore quite important to have some rigorous restrictions on $\partial(H^*, \pi^*)$: such restrictions could, for instance, be provided by conservation laws. Some examples on this point can be found in ([14]).

5. – The scaling laws.

Here we rapidly derive the scaling laws at $h = 0$. We show that all the critical exponents can be expressed in terms of the two constants $\varrho = 1 + (2 - \eta)/d$ (see ([6])) and $(\ln_2 \lambda)/w$.

The following arguments are mainly heuristic since the number of assumptions made so far is so large that, anyhow, we have lost control of the errors. We apply to (4.5) K_ϱ^l and we obtain

$$K_\varrho^{n+l}(\beta H_0, \pi_0) = K_\varrho^{n+l}(\beta_c H_0, \pi_0) + (\beta - \beta_c)^w \mathscr{L}^l \Delta_n, \tag{5.1}$$

where $\mathscr{L}$ is the operator K_ϱ linearized around $K_\varrho^n(\beta_c H_0, \pi_0) \simeq (H^*, \pi^*)$. Therefore using III) we interpret the value $\bar{l}$, such that $(\beta - \beta_c)^w \lambda^{\bar{l}} = 1$, as linked to the correlation length $L(\beta)$ by

$$L(\beta) \propto 2^{\bar{l}} \propto (\beta - \beta_c)^{-w/\ln_2 \lambda}, \tag{5.2}$$

where the symbol $\propto$ is supposed to mean asymptotically proportional in the limit $\beta \to \beta_c^-$. So the exponent known as ν is

$$\nu = \frac{w}{\ln_2 \lambda}. \tag{5.3}$$

We have already remarked that the exponent known as η is given by $\varrho = 1 + (2 - \eta)/\delta$ (cf. footnote ([12])). Let us compute one more exponent: consider the exponent called α in the literature ([4]). The starting point is (3.15) written

([13]) K. G. WILSON and M. E. FISHER: *Phys. Rev. Lett.*, **28**, 240 (1972); M. E. FISHER: *Phys. Rev. Lett.*, **29**, 917 (1972).

([14]) G. GALLAVOTTI, H. J. F. KNOPS and H. VAN BEYEREN: preprint (1974).

for $l = \bar{l} - \xi$ with ξ an arbitrary large constant and $\bar{l}$ given by (5.2). We find

$$(5.4)\qquad f(\beta H, \pi) = \sum_{k=1}^{\bar{l}-\xi} \frac{f\big((\beta H_{k-1}, \pi_{k-1})_0\big)}{2^{dk}} + 2^{-d(\bar{l}-\xi)} f(\beta H_{\bar{l}-\xi}, \pi_{\bar{l}-\xi}) =$$
$$= \sum_{k=1}^{\bar{l}-\xi} f\big((\beta H_{k-1}, \pi_{k-1})_0\big) - f\big((\beta_c H_{k-1}, \pi_{k-1})_0\big) +$$
$$+ \sum_{k=1}^{\bar{l}-\xi} \frac{f\big((\beta_c H_{k-1}, \pi_{k-1})_0\big)}{2^{dk}} + 2^{-d(\bar{l}-\xi)} f(\beta H_{\bar{l}-\xi}, \pi_{\bar{l}-\xi}) .$$

Using (5.1) and assuming that the free energy of the constrained model $(\beta H_{k-1}, \pi_{k-1})_0$ is a Hölder continuous function of the Hamiltonian parameters in the neighbourhood of $(\beta_c H_{k-1}, \pi_{k-1})_0$ and if we define its Hölder exponent as $2 - \alpha_0$, we find that (see (5.2), (5.3))

$$(5.5)\qquad f(\beta H, \pi) \underset{\beta \to \beta_c}{\simeq} (\beta - \beta_c)^{d\nu} F + G \sum_{k=1}^{\bar{l}-\xi} \frac{[(\beta - \beta_c)^w \lambda^k]^{2-\alpha_0}}{2^{dk}},$$

where F and G are some constants.

If one assumes $\lambda < 2^d$ and either $w = 1$, $\alpha_0 = 1$ [15] or some suitable relations between λ, w, α_0, it follows that

$$(5.6)\qquad f(\beta H, \pi) \simeq F(\beta - \beta_c)^{d\nu},$$

which in terms of the usual definitions in the literature reads

$$d\nu = 2 - \alpha .$$

However, if one does not make the above assumptions, many other results are possible including an explicit dependence of α on the parameter w, which would then become a third independent parameter.

6. – The hierarchical models.

This is a class of models [15,16] where the assumptions 1), 2), I)-IV) can be rigorously checked [2]. However these models have long-range periodic interactions. Roughly the hierarchical models are characterized by the property that there is one value of ϱ, ϱ', such that

$$(6.1)\qquad K_{\varrho'}(\beta H, \pi) = (\beta H, \pi'),$$

([15]) This would be more natural if the constrained Hamiltonian $(H_l, \pi_l)_0$ were never critical or had, in some sense, an ∞-order transition (cf. Sect. **4**, discussion of assumption I)).

([16]) F. J. Dyson: *Comm. Math. Phys.*, **12**, 91, 212 (1969); **21**, 269 (1971).

i.e. the « interaction » part of the Hamiltonian is invariant under $K_{\varrho'}$ [2]. We shall not need in the sequel an explicit expression of H, which the interested reader can find for instance in ref. [2,16,17].

Among the hierarchical models which give rise to the same ϱ' in (6.1) there are particular models (*i.e.* particular π's) which have a critical point β_c and, at β_c, assumption II) is verified with $\varrho = \varrho'$.

Let us denote by $H_{\varrho'}$ a hierarchical Hamiltonian which in (6.1) gives rise to a certain value ϱ'. BLEHER and SINAI have been able to study the critical behaviour of classes of models of the form $(\beta H_\varrho, \pi)$ in a range of the parameter ϱ between $1 < \varrho < \frac{3}{2} + \varepsilon$, where ε is some positive number [2]. They were not only able to prove 1), 2), I)-IV) in precise mathematical terms but also to go often beyond these assumptions and obtain a very detailed picture of the neighbourhood of the critical point for such models [18].

We note that, for a hierarchical model with index ϱ, the recursion formula for the block spins is simply an integral operator on the « free » part of the Hamiltonian

$$\pi'(\nu_x) = \frac{\displaystyle\int \prod_{\xi\in\{x,1\}} d\sigma_\xi\, \pi(\sigma_\xi) \exp[-\beta H^{(1)}(\underline{\sigma}_x)]\, \delta\left(\frac{\sum_\xi \sigma_\xi}{2^{d\varrho/2}} - \nu_x\right)}{\text{normalization}}; \tag{6.2}$$

we call $\tilde{K}_{\varrho,\beta}$ the operator defined by (6.2). This equation under mild conditions on $H^{(1)}$ always has a Gaussian fixed point which is usually called the « simplest » or « trivial » fixed point.

7. – The ε-expansion.

The transformation K_ϱ seems to be hopelessly complicated in the general case, particularly when one comes to the discussion of the fixed point (H^*, π^*) and its tangent space $\partial(H^*, \pi^*)$ which is relevant for a particular initial model $(\beta H, \pi)$. To deal with such question in, say, the Ising model one needs some drastic approximations whose physical meaning is not always clear.

There is, however, an interesting class of models, the hierarchical models, for which it is possible to develop an approximation scheme which seems to be under control.

The approximation scheme is the famous ε-expansion, which we discuss heuristically in the following pages and which has recently proven to be rigorous for a certain class of hierarchical models [18].

It is assumed that the hierarchical model $(\beta H_\varrho, \pi)$, see preceding Section,

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(17) G. A. BAKER jr.: *Phys. Rev. B*, **5**, 2622 (1972).
(18) JA. G. SINAI: preprint.

has a critical point which is described as long as possible by the simplest Gaussian fixed point π_g^* of eq. (6.2), together with the space $\partial(H^*, \pi^*)$ consisting of the functions of the form $\pi_g^*(\nu)f(\nu)$, $f \in L_1(\pi_g^*(\nu)\,d\nu)$ or a subspace of it prescribed by *a priori* arguments [14]. « As long as possible » means as long as property IV) holds.

It can be checked that the spectrum of the operator obtained by linearizing $\tilde{K}_\varrho$ around π_g^* has the spectrum (independent of $H^{(1)}$ and β)

$$\lambda_n^\varrho = 2^d (2^{d(\varrho-2)})^n\,, \qquad n = 1, 2, \dots, \tag{7.1}$$

and the associated eigenfunctions are polynomials of degree $2n$.

So, unless some conservation laws force $\partial(H^*, \pi^*)$ to be orthogonal to the eigenspace of λ_2, the critical point cannot be described by Wilson's theory around the above fixed point if

$$\varrho > \tfrac{3}{2}\,. \tag{7.2}$$

When $\varrho > \frac{3}{2}$ one assumes that there is another fixed point of the form

$$\pi^*(\nu) = \pi_g^*(\nu)\exp[a\nu^2 + b\nu^4 + O(\varepsilon^3\gamma^6)]\,, \tag{7.3}$$

where a, b are assumed to be of order $\varepsilon = \varrho - \frac{3}{2}$, which describes, together with the linear space of the functions of the form $\pi^*(\nu)f(\nu)$, $f(\nu) \in L_1(\pi^*(\nu)\,d\nu)$, the theory of the hierarchical model (unless *a priori* reasons forbid it, as in the Gaussian case; see also ref. [14]).

It is easy to show, consistently to order ε, that such a fixed point indeed exists and, if $\varrho - \frac{3}{2} = \varepsilon > 0$, has the right spectral properties.

Applying $\tilde{K}_\varrho$ to (7.3) one gets

$$\pi'(\nu_x) = c\int \exp[-B(\underline{\nu}, \underline{\nu})]\,\delta\left(\frac{\sum_{i\in\{x,1\}} \nu_i}{2^{(\varrho/2)d}} - \nu_x\right)\prod_{i=1}^{2^d}\left(\pi_g^*(\nu_i)\exp[a\nu_i^2 + b\nu_i^4]\,d\nu_i\right), \tag{7.4}$$

where $B(\underline{\nu}, \underline{\nu})$ is the Hamiltonian restricted to a box of side 2^d. Introducing orthogonal co-ordinates $\bar{\nu}, y_1, \dots, y_{2^d-1}$ such that

$$\left\{\begin{aligned} \nu_i &= \frac{\bar{\nu}}{2^{d/2}} + \sum_{J=1}^{2^d-1} \alpha_{iJ}\,y_J\,, \\ \bar{\nu} &= \sum_{i=1}^{2^d} \frac{\nu_i}{2^{d/2}}\,, \end{aligned}\right. \tag{7.5}$$

furthermore, we assume that B is negative definite and that ([19])

$$B(\underline{\nu}, \underline{\nu}) = -\alpha_0 \bar{\nu}^2 + \hat{\beta}(\underline{y}, \underline{y}) . \tag{7.6}$$

If one retains only terms up to order ε^2, one obtains that the constants a and b become a', b' such that

$$\begin{cases} a' = 2^d \zeta^2 a + 6\zeta^2 \langle y^2 \rangle b + b^2 \alpha + O(\varepsilon^3) , \\ b' = 2^d \zeta^4 b + 18\zeta^4(\langle y^4 \rangle - \langle y^2 \rangle^2) b^2 + O(\varepsilon^3) , \end{cases} \tag{7.7}$$

where

$$\begin{cases} \zeta = 2^{d(\varrho-2)/2} , \\ \langle y^{2n} \rangle = \dfrac{\displaystyle\int \prod_{i=1}^{2^d-1} (\pi_\varrho^*(y_i) \exp[a y_i^2] \mathrm{d}y_i) \exp[\hat{\beta}(\underline{y}, \underline{y})] \sum_{i=1}^{2^d-1} y_i^{2n}}{\displaystyle\int \prod_{i=1}^{2^d-1} (\pi_\varrho^*(y_i) \exp[a y_i^2] \mathrm{d}y_i) \exp[\hat{\beta}(\underline{y}, \underline{y})]} , \end{cases} \tag{7.8}$$

and α is a known function of more complicated averages of the form (7.8).
This leads to the fixed point

$$\begin{cases} \bar{a} = \dfrac{6\zeta^2}{1 - \zeta^2 2^d} \langle y^2 \rangle_{a=0} \bar{b} , \\ \bar{b} = -\dfrac{2^d \zeta^4 - 1}{18\zeta^4(\langle y^4 \rangle_{a=0} - \langle y^2 \rangle_{a=0}^2)} , \end{cases} \tag{7.9}$$

where, indeed, $\bar{a}$, $\bar{b}$ are $O(\varepsilon)$ with $\varepsilon = 2^d \zeta^4 - 1$. The operator $\tilde{K}_\varrho$ linearized around the fixed point $\pi^*(\nu)$ can be studied to first order in ε on the 2-parameter space (a, b) and one easily finds that the eigenvalues of the linearized operator are

$$\begin{cases} \lambda_1 = 2^{d/2}(1 + \varepsilon/6) + O(\varepsilon^2) , \\ \lambda_2 = 1 - \varepsilon + O(\varepsilon^2) . \end{cases} \tag{7.10}$$

The linearization around the trivial fixed point, which to order ε can be identi-

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([19]) Notice that if $B(\underline{\nu}, \underline{\nu})$ is a bilinear form in the ν_i's a sufficient condition for (7.6) is given by $\sum_{J=1}^{2^d} B_{iJ} = \alpha_0$ (independent of i!), which is a condition of symmetry saying that all the spins in the same block enter symmetrically in the Hamiltonian. The conditions on the signs of the eigenvalues of B is a kind of attractiveness condition which is needed also to avoid divergences in (7.8).

fied with the choice $a = b = 0$, gives of course

$$(7.11) \qquad \begin{cases} \lambda_1^g = 2^{d/2}(1 + \varepsilon/2)\,, \\ \lambda_2^g = 1 + \varepsilon\,. \end{cases}$$

So the ε-expansion provides a tool for estimating the change of λ_1 as a function of ε in the hierarchical model.

8. – Conclusions.

We have tried to explore the basic assumptions of Wilson's theory of the Kadanoff renormalization group in order to point out explicitly a number of conjectures which seem very interesting for the mathematical physicist.

We hope that the above very heuristic and, sometimes, puzzling discussion may help in understanding the relations between the beautiful intuitions of KADANOFF and WILSON and the modern theory of the limit theorems in probability theory.

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RIASSUNTO

Questo lavoro contiene essenzialmente un'esposizione della teoria di Wilson del gruppo di rinormalizzazione di Kadanoff.

Закон Лавуазье и критическая точка.

Резюме (*). — В этой работе рассматривается теория Вильсона для группы перенормировки Каданова.

(*) *Переведено редакцией.*