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Matematica. – *Zeta functions and basic sets. Nota di* GIOVANNI GALLAVOTTI*, presented*(*) *by the Member* B. SEGRE.

SUMMARY. - We give an example of a smooth flow with a basic set whose zeta-function is not meromorphic.

§1. Introduction

Zeta functions considered here are functions of one complex variable arising from the problem of studying the distribution of the periods of the periodic orbits generated on a manifold M by the action of groups of maps homomorphic to $\mathbb{R}$ or $\mathbb{N}$ (the continuous and discrete, respectively, cases).

The functions describe the distribution of the periodic orbits periods in the same sense in which the Riemann zeta function describes the distribution of the logarithms of the prime numbers.

In the case in which the group of maps is a group of diffeomorphisms of a Riemannian manifold homomorphic to $\mathbb{N}$ and, furthermore, the group satisfies the Axiom A (Cf. Appendix 2), it can be shown that the associated zeta function is meromorphic.

Our purpose here is to show, by providing an example, that such property does not extend, in general, to the case of a group of maps which, although satisfying the axiom A, is homomorphic to $\mathbb{R}$.

Since we propose an example this work is of a very technical nature and we shall suppose that the reader is already familiar with the problem and ts interest (see for this purpose the paper [1], Chap. I, §1,2,3 and Chap. II, §1)

§2. Formulation of the problem

In studying general properties of periodic orbits with respect to groups of diffeomorphisms homomorphic to $\mathbb{N}$ or $\mathbb{R}$ the analysis of the properties of the "zeta functions" associated with them, [1].

Let M be a compact Riemannian manifold and let $S_t : M \longleftrightarrow M,\ t \in \mathbb{R}$ be a C^r-flow ($1 \le r \le \infty$), on M (which, obviously will be assumed to be of class C^r at least).

Suppose that the set Ω of the "non wandering points", [1], contains a "basic set" Λ, *i.e.* a closed, S_t-invariant $\forall t \in \mathbb{R}$ set such that

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(i) Λ does not contain fixed points for S_t and all $t \in \mathbb{R}$, and it is a hyperbolic set for the flow;

(ii) periodic orbits contained in Λ are dense in Λ;

(iii) Λ contains a dense orbit;

(iv) There is an open set $U \supset \Lambda$ such that $\Lambda = \cap_{t\in\mathbb{R}} S_t U$.

We shall associate with each periodic orbit $\gamma \subset \Lambda$ its period $T(\gamma)$. The set of the periodic orbits of the flow will be denoted $\mathrm{op}(S_t)_{t\in\mathbb{R}}$.

If Λ is a basic set the zeta function for the restriction of the flow to Λ is defined by the infinite product

$$(*) \qquad \zeta(s) = \prod_{\substack{\gamma\in \mathrm{op}(S_t)_{t\in\mathbb{R}} \\ \gamma\subset\Lambda}} (1 - e^{-sT(\gamma)})^{-1}$$

which can be shown to be convergent for $\mathrm{Re}\,(s)$ large enough, [1,3].

There exist flows whose basic sets have a meromorphic zeta function (*i.e.* $(*)$ admits a meromorphic extension to $\mathbb{C}$), [3].

The just given definitions can be given, by natural analogy, also in the case in which the group $(S_t)_{t\in\mathbb{R}}$ is replaced by a group $(S^n)_{n\in\mathbb{N}}$ homomorphic to the group of the integers. The extensions are obvious, [1], except that in defining basic sets the condition of absence of fixed points is dropped. Thus, for instance, if $\Lambda \subset M$ is a basic set for the group $(S^n)_{n\in\mathbb{N}}$ of diffeomorphisms of M, let

$$(**) \qquad \zeta(s) = \prod_{\substack{\gamma\in \mathrm{op}(^n)_{n\in\mathbb{N}} \\ \gamma\subset\Lambda}} (1 - e^{-sT(\gamma)})^{-1}$$

where now $T(\gamma)$ is an integer; it is possible to prove that the infinite product $(**)$ converges for $\mathrm{Re}\,(s)$ large enough, [3].

It is proved that the zeta function associated with a basic set of a group $(S^n)_{n\in\mathbb{N}}$ is always a meromorphic function, [4,3].

The main result of the present work is the construction of a $\mathbb{C}^r$–flow with a basic set whose zeta function is not meromorphic ($1 \le r < \infty$). Such example can be used to construct a $\mathbb{C}^r$–flow satisfying the axiom A and whose zeta function cannot be extended to a function meromorphic on $\mathbb{C}$: see the final remark in appendix 2 below.

In the following we shall use the following observation: in general one can try to define, via $(*)$ and $(**)$, the zeta function associated with a group of maps, homomorphic to $\mathbb{R}$ or $\mathbb{N}$, of an abstract space K into itself and with a set, invariant with respect to the group action, $L \subset K$ however the convergence abscissa for the infinite product will, very often, be infinite. In the following, when considering zeta functions associated with invariant sets, we shall always suppose, whenever this is not implied by the assumptions, that the considered infinite product converges for $\mathrm{Re}\,(s)$ large enough.

§3. Generalized zeta functions

In the construction of our example it is useful to consider a generalization of $(**)$, [3,5].

Let $(S^n)_{n\in\mathbb{N}}$ be a group of maps of a space M homomorphic to $\mathbb{N}$; let 4Λ be an invariant set under the action of the group. Let $\mathrm{op}(S^n)_{n\in\mathbb{N}}$ be the set of the periodic orbits γ and let $T(\gamma)$ be the period of γ. Let $f : M \to \mathbb{R}$ be a function defined on M; define the "F–weighted period" of $\gamma \in \mathrm{op}(S^n)_{n\in\mathbb{N}}$ as

$$T_f(\gamma) = \sum_{j=0}^{T(\gamma)} f(S^j x)$$

where x is an arbitrary point of γ.

The zeta function with weight f relative to the set Λ and to the given group of maps of M is defined by

$$\zeta(s,f) = \prod_{\gamma\in \mathrm{op}(S^n)_{n\in\mathbb{N}}} (1-e^{-sT_f(\gamma)})^{-1}$$

It is useful, for an alternative expression of zeta, to introduce the set $\widetilde{\mathrm{op}}(S^n)_{n\in\mathbb{N}}$ of the pairs $\widetilde{\gamma} = (\gamma, k)$ where $\gamma \in \mathrm{op}(S^n)_{n\in\mathbb{N}}$ and $k = 1,2,3,\ldots \in \mathbb{N}^+$. This is the set of the "periodic orbits with period which is not necessarily minimal." We shall set

$$T(\widetilde{\gamma}) = T(\gamma,k) = kT(\gamma);\quad T_f(\widetilde{\gamma}) = t_f(\gamma,k) = kT_f(\gamma)$$

$$\text{and}\ \ \widetilde{\gamma}\subset\Lambda \qquad \text{if} \qquad \widetilde{\gamma} = (\gamma,k),\ \gamma\subset\Lambda$$

Simple combinatorial considerations show that

$$\zeta(s,f) = \exp \sum_{\widetilde{\gamma}\in\widetilde{\mathrm{op}}(S^n)_{n\in\mathbb{N}}} e^{-s\,T_f(\widetilde{\gamma})}.$$

The interest of the above further generalizations of the zeta functions resides in the following easy Lemma, [5]:

LEMMA. *Let M be a space on which acts a group $(S^n)_{n\in\mathbb{N}}$ of maps leaving a set $\Lambda\subset M$ invariant. Let $f : M \to \mathbb{R}$ be a positive function on M. Let $(S_t)_{t\in\mathbb{R}}$ be the "special flow" (see appendix (2) built upon Λ by a "ceiling function" f and the given group of maps.*
Let $\zeta(s)$ be the zeta function associated with the special flow and let $\zeta(s,f)$ the zeta function associated with set Λ and the group $(S^n)_{n\in\mathbb{N}}$; then $\zeta(s) = \zeta(s,f)$ if $\mathrm{Re}\,(s)$ is large provided the infinite products defining $\zeta(s)$ and $\zeta(s,f)$ converge for $\mathrm{Re}\,(s)$ large enough.

The Lemma will be used together with the following theorem, [2]:

THEOREM. *Let S be the translation by one unit to the right on the space K of the bilateral sequences of symbols 0 or 1 and let f be a Hölder continuous positive function on K.*[1]
The special flow on K, defined by S and f, can be immersed as a basic set in a C^r–flow, $1\le r<\infty$, on a finite dimensional, compact Riemannian manifold.

By the above two results the construction of our example can proceed via the construction of a non negative Hölder continuous function on K with a generalized zeta function, with respect to the translation S cannot be extended to a meromorphic function on $\mathbb{C}$.

§4 Certain Hölder continuous functions and their zeta function

Let $K = \prod_{i\in\mathbb{N}}\{0,1\}$ = (space of the bilateral sequences of symbols 0 and 1) regarded as a metric space with the metric

$$d_a(\{\xi_i\}_{i=-\infty}^{+\infty}, \{\eta_i\}_{i=-\infty}^{+\infty} = a^p$$

where p = maximum k among those for which $\xi_j = \eta_j$ if $|j|\le k$ and a is a prefixed number in $(0,1)$.

A function f on K is called Hölder continuous, [2], on K if there is $a\in(0,1)$ such that $\sup_{x,y\in K}\frac{|f(x)-f(y)|}{d_a(x,y)} < +\infty$. It is easy to construct examples of Hölder continuous functions on K; we shall encounter interesting ones below.

[1] See §4 for a definition.

Consider the elements of K as subsets of $\mathbb{N}$: a point $x = \{\xi_i\}_{i=-\infty}^{\infty} \in K$ will be identified with he set $X(x) = \{i | \, i \in \mathbb{N}, \xi_i = 1\}$. Let Φ be a function defined on the finite subsets of $\mathbb{N}$ and such that there exists a $\kappa > 0$ for which:

(i) $\sum_{\{0\}\subset X} |\Phi(X)| e^{\kappa (diameter X)} < +\infty$
(ii) $\Phi(X) = \Phi(Y)$ if X and Y are congruent

Define, on the subsets of $\mathbb{N}$ (finite or not) the function A_Φ:

$$A_\Phi(X) = \sum_{\{0\}\subset L \subset X} \frac{\Phi(L)}{|L|}$$

where $|L|$ = number of points in L, and the sum is over L among the finite subsets of X. By the adopted identification between points in K and subsets of $\mathbb{N}$ the function A_Φ can be regarded as a function defined on K. It is easy to see that A_Φ is Hölder continuous on K.

We shall consider the following special case ("Fisher's model", [7]):

$$\Phi(L) = 0 \;\; \text{unless} \;\; |L| = (\text{diameter of } L) + 1$$

If, instead, $L = (i, i+1, \ldots, i+j-1)$ for some pair of integers i, j we shall set

$$\Phi(L) = \varphi_j.$$

The model is a well known "machine" to build examples and counter-examples in the theory of phase transitions [7,8].

Let $G > 0$ such that $\min(G + A_\Phi$ and let $f = G + A_\Phi$ and, furthermore,

$$W_\ell = \ell \sum_{i=1}^{\ell} \varphi_i - \sum_{i=1}^{\ell-1} i\varphi_{i+1}, \qquad \ell \in \mathbb{N}^+$$

$$\Phi_0 = \sum_{i=0}^{\infty}; \quad \Phi_1 = \sum_{i=1}^{\infty} i\varphi_i.$$

Lemma. *The function $\zeta(s,f)$ is given by*

$$\zeta(s,f)^{-1} = (1 - e^{-sG})(1 - e^{-sG - s\Phi_0})(1 - F(s))$$

where

$$F(s) = \frac{e^{-sG + s\Phi_1}}{1 - e^{-sG}} \sum_{\ell=1}^{\omega} e^{-\ell\, s\, G} e^{-sW_\ell}.$$

The proof of this lemma is a computation which proceeds, without any conceptual difficulty, along the lines of reference [7], see the appendix below for a concise exposition of the technique of [7]. If $(\varphi_j)_{j=1}^{\infty}$ is defined by

$$\varphi_1 \text{ arbitrary}, \quad \varphi_n = x_n - 2x_{n-1} + x_{n-2}, \qquad n \geq 2, \; \textit{where}$$
$$x_n = \log(1 + \frac{\ell^n}{n}) \qquad n \geq 1, \; \lambda < 1, \; ; x_0 = 0$$

It follows that: $W_n = n\Phi_0 - \Phi_1 - \log(1 + \frac{\ell^n}{n})$.

The example is the constructed since the function

$$\sum_{n=1}^{\infty} e^{-(G+\Phi_0)m} (1 + \frac{\ell^n}{n})^s$$

cannot be extended to a meromorphic function as it presents a non polar singularity in $s_0 = \frac{\log \ell}{G+\Phi_0} < 0$. Remark also that in $s = 0$ there is no singularity for $\zeta(s,f)$ and, in fact, $\zeta(0,f) = -1$.

Appendix I. Proof of the lemma in Sec.4

This appendix contains a proof of the lemma in Sec.4 through via a simple adaptation of [7]. We shall first compute, for $\sigma \in \mathbb{C}$

$$\Delta(\xi) = \sum_{p=1}^{\infty} \frac{1}{p} \xi^p \sum_{S^p x = x} e^{-T_{A_\Phi}(x)}.$$

The points $x \in K$ such that $S^p x = x$ can be identified, (as explained in §3) with the periodic subsets of $\mathbb{N}$ with period p or, equivalently, with the subsets of set $\{1, 2, \dots, p\} \subset \mathbb{N}$.

Selecting the second of the just described alternatives, we shall call a subset $X \subset \{1, 2, \dots, p\} \subset \mathbb{N}$ a "configuration". If we imagine to group the sites in $\{1, 2, \dots, p\}/X$, *i.e.* "empty sites", into connected groups and if, likewise, the points in x, "occupied sites", are grouped into connected sets it is possible to describe a configuration by a sequence of integers according to the following six possibilities:

$$(I) \qquad (v_1, o_1, \dots, v_k, o_k, v_{k+1}),\ o_i, v_i > 0\ \forall\, i,$$

and $v_1, \dots, v_{k+1}$ denote the numbers of empty sites in the various connected groups of sites numbered from left to right; $k \ge 1$ and

$$\sum_{i=1}^{k+1} v_i + \sum_{i=1}^{k} o_i = p.$$

Continuing the case by case analysis, with similar notations, the other possibilities are:

$$\begin{aligned}
(II) &\qquad (v_1, o_1, \dots, v_k, o_k),\ o_i, v_i > 0\ \forall\, i,\ k \ge 1,\ \sum_{i=1}^{k} v_i + \sum_{i=1}^{k} o_i = p \\
(III) &\qquad (o_1, v_2, \dots, , o_k, v_{k+1}),\ o_i, v_i > 1\ \forall\, i,\ k \ge 1,\ \sum_{i=2}^{k+1} v_i + \sum_{i=1}^{k} o_i = p \\
(IV) &\qquad (o_1, v_2, \dots, v_k, o_k),\ o_i, v_i > 1\ \forall\, i,\ k \ge 2,\ \sum_{i=2}^{k} v_i + \sum_{i=1}^{k} o_i = p \\
(V) &\qquad o_1 = p \\
(VI) &\qquad v_1 = p
\end{aligned}$$

If X is of the type I, II, III and if $x \in K$ is the point of K such that $S^p x = x$ associate with it via the described correspondence we find

$$T_{A_\Phi}(x) = \sum_{j=1}^{k} W_{o_j}$$

where

$$W_n = n\varphi_1 + (n-1)\varphi_2 + \dots + (n-n+1)\varphi_n$$

Whereas if X is of the type (IV) it is: $T_{A_\Phi}(x) = \sum_{j=0}^{k-1} W_{O_j} + W_{o_1+o_k}$.
If X is of the type (V) it is: $T_{A_\Phi}(x) = N \sum_{j=0}^{\infty} \varphi_j$.
If X is of the type (VI), finally: $T_{A_\Phi}(x) = 0$.

It is now easy to compute, by summation of a few geometric series, the function $\xi\Delta'(\xi)$ and, therefore, $\Delta(\xi)$. The result is that $\xi\Delta'(\xi)$ can be written as:

$$
\begin{aligned}
&-\xi\frac{d}{d\xi}\log(1-\xi)-\xi\frac{d}{d\xi}\log(1-\xi e^{-\Phi_0})-\\
&-\xi\frac{d}{d\xi}\log(1-\frac{\xi}{1-\xi}\sum_{\ell=1}\xi^\ell e^{-W_\ell})
\end{aligned}
$$

and from the latter formula the proof f the lemma follows immediately.

Appendix 2. Some definitions

Definition 1. *Wandering and non-wandering points.*

Let $(S_t)_{t\in\mathbb{R}}$ (or $(S^n)_{n\in\mathbb{N}}$ be a group of invertible maps defined and continuous on a topological space M. A point $x\in M$ is said "wandering" if there is a vicinity U of x such that $U\cap S_t(U)$ for all t with $|t|$ large enough (respectively $U\cap S^n(U)$ for all $|n|$ large enough). It is manifest that the set of wandering points is open, hence the set of non-wandering points is closed.

Definition 2. *Hyperbolicity of a set (flow case).*

Let $(S_t)_{t\in\mathbb{R}}$ be a group of invertible maps defined and continuous on a compact Riemannian manifold. An invariant set $\Lambda\subset M$ is "hyperbolic" if for all $x\in\Lambda$ it is possible to decompose the space T_xM tangent to M in x as:

$$T_xM=V_x^{(s)}\oplus V_x^{(i)}\oplus V_x$$

in three subspaces depending continuously on $x\in\Lambda$ and such that

(1) $dS_t\,:\,V_x^{(s)}\longleftrightarrow V_{S_tx}^{(s)}\,;\ dS_t\,:\,V_x^{(i)}\longleftrightarrow V_{S_tx}^{(i)}\,;\ dS_t\,:\,V_x\longleftrightarrow V_{S_tx}\,;$

(2) There exist $a,b>0$ such that:

$$||dS_t||_s\le ae^{-bt},\ t>0;\ ||dS_t||_I\le ae^{bt},\ t<0;\ ||dS_t||_0\le a,\ \forall t$$

where $||dS_t||_s,||dS_t||_i,||dS_t||_0$ denote the norms of the restrictions of dS_t to the spaces $V_x^{(s)},V_x^{(i)},V_x$ respectively.

(3) Dimension of $V_x\ \equiv\ 1$.

Definition 3. *Hyperbolicity of a set (discrete case).*

Let $(S^n)_{n\in\mathbb{N}}$ be a group of diffeomorphisms on a compact Riemannian manifold. An invariant set $\Lambda\subset M$ is "hyperbolic" if for all $x\in\Lambda$ it is possible to decompose the space T_xM tangent to M in x as:

$$T_xM=V_x^{(s)}\oplus V_x^{(i)}$$

in two subspaces depending continuously on $x\in\Lambda$ and such that

(1) $dS^n\,:\,V_x^{(s)}\longleftrightarrow V_{S_tx}^{(s)}\,;\ dS^n\,:\,V_x^{(i)}\longleftrightarrow V_{S_tx}^{(i)},\ \forall n\in\mathbb{N}$

(2) With notations similar to the one of definition 2, there exist $a,b>0$ such that:

$$||dS^n||_s\le ae^{-bn},\ n>0;\ ||dS^n||_i\le ae^{bn},\ n<0;\ ||dS_t||_0\le a,\ \forall t$$

Definition 4. *Axiom A (discrete case).*

Let $(S^n)_{n\in\mathbb{N}}$ be a group of diffeomorphisms on a compact Riemannian manifold. It will be said the satisfy axiom A if:

(1) the set Ω of the non-wandering points is hyperbolic.

(2) The set of periodic orbits is dense in Ω.

DEFINITION 5. *Axiom A (flow case).*

Let $(S_t)_{t\in\mathbb{R}}$ be a group of diffeomorphisms on a compact Riemannian manifold. It will be said the satisfy axiom A if:

(1) the set Ω of the non-wandering points is hyperbolic and contains a finite number of fixed points disjoint from the remaining part of Ω.

(2) The set of periodic orbits is dense in Ω.

DEFINITION 6. *Special flows.*

Let $(S^n)_{n\in\mathbb{N}}$ be a group of diffeomorphisms on a compact Riemannian manifold. Let f be a strictly positive function on Λ.Consider the space $\widetilde{\Lambda}_f$ of the pairs (x,m) with $m\in\Lambda$ and $0\le m\le f(x)$. Let Λ_f be the space obtained by identifying the points $(x,f(x))$ and $(S(x),0)$ of $\widetilde{\Lambda}_f$, $x\in\Lambda$. Define a group of maps $(S_t)_{t\in\mathbb{R}}$ of Λ_f as $(t\ge 0)$

$$\begin{aligned}
S_t(x,m)=&(x,m+t) \qquad \text{if } 0\le m\le f(x)\\
S_t(x,m)=&(S(x),m+t-f(x)) \qquad \text{if } f(x)\le m+t\le f(x)+f(S(x))\\
S_t(x,m)=&(S^2(x),m+t-f(x)-f(S(x))) \qquad \text{if}\\
&\qquad f(x)+f(S(x))\le m+t\le\ 0\le m\le f(x)+f(S(x))+f(S^2(x))
\end{aligned}$$

etc.; furthermore let $S_t=(S_{-t})^{-1}$ for $t\le 0$.
The latter group of maps of Λ_f is called the "special flow" built upon Λ via $(S^n)_{n\in\mathbb{N}}$ and the "ceiling function" f.

To understand the interest of the notion of basic set it is convenient to quote the following theorem, [1]:

THEOREM. *If $(S^n)_{n\in\mathbb{N}}$ is a group of diffeomorphisms of a compact Riemannian manifold M and $(S^n)_{n\in\mathbb{N}}$ satisfies axiom A then the set Ω of non-wandering points can be decomposed in a finite number of pairwise disjoint basic sets.*

The obvious version of the statement for flows satisfying axiom A is also true, [1].

REMARK. The C^r-immersion theorem in §3 of a special flow built via a Hölder continuous function upon the space of the sequences of symbols 0 and 1 on which acts the group of translations, can be strengthened so that the image of the special flow consists, together with two more isolated fixed points, the set of the non-wandering points of a C^r-flow on a compact Riemannian manifold. This follows from the proof in [2] and from the considerations on the construction of the "horseshoe" in [1].
From the latter remark we see how the example discussed in the present paper also provides an example of a flow satisfying axiom A whose zeta function cannot be extended to a meromorphic function.

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