One Dimensional Lattice Gases with Rapidly Decreasing Interaction

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Abstract

We consider the pressure and the correlation functions of a one dimensional lattice gas in which the mutual interaction decreases as $r\exp(-n^t)$, $(r, t>0)$, when the interparticle distance $n\to\infty$. We prove that such a system cannot show phase transitions of order $k\geq 1$ in the sense that the pressure and the correlation functions are infinitely differentiable with respect to any relevant parameter (such as the temperature or the chemical potential).

§ 1. Introduction

It has been conjectured [1] that, for one dimensional lattice gases, the existence of the k^{th} moment of the potential excludes the possibility of phase transitions of order lower than k, *i.e.* the pressure should be k-times continuously differentiable with respect to the chemical potential or the temperature or other relevant variables. This conjecture has been proved for $k=0, 1$ [2, 3, 4, 5], and it has also been shown that at least for $k=0, 1$ this result is the strongest possible since there are examples in which the first moment is divergent and the first order derivative of the pressure with respect to the chemical potential (or the temperature) does not exist everywhere [6, 7, 8].

In this paper we examine the case in which the interaction is, roughly speaking, decreasing at least like $r\exp(-n^t)$ $(r, t>0$, n denotes the interparticle distance) so that all the moments are convergent, and we prove that the pressure as well as the correlation functions are infinitely differentiable.

§ 2. Notations and Previous Results

Let us consider a one dimensional lattice $Z=\{\dots, -1, 0, 1, 2, \dots\}$ and suppose that if $N(X)$ particles occupy the set $X\subset Z$, their interaction energy is given by

$$U_\Phi(X)=\sum_{S\subset X}\Phi(S) \tag{2.1}$$

where $\Phi(\cdot)$ is the (in general many-body) interaction potential and the sum runs over the subsets of X. We require Φ to be translationally invariant (*i.e.* $\Phi(X)=\Phi(\tau_n X)$ where $\tau_n X=\{x+n: x\in X\}$), and we suppose that the potential Φ is rapidly decreasing, *i.e.*

$$\sum_{0\in S\not\subset[-n,n]}(\operatorname{diam} S)\,|\Phi(S)|<r\exp(-n^t) \tag{2.2}$$

where diam S denotes the largest distance between points in S. In the case the interaction $\Phi(S)$ has only one body (chemical potential) and two body terms (2.2) becomes

$$\sum_{k>n} k\,|\Phi(k)| < r\exp(-n^t)$$

where $\Phi(k)=\Phi(x, x+k)$ denotes the two body potential.

We shall call $\mathscr{B}$ the (linear) space of the potentials satisfying (2.2). If $\Phi\in\mathscr{B}$, then

$$\|\Phi\| = \sum_{0\in S} (\operatorname{diam} S)\,|\Phi(S)| < \infty. \tag{2.3}$$

We shall prove the following theorem.

Theorem 1. *If $Z_n(\Phi)$ denotes the partition function associated with an interval of length n, then the thermodynamic pressure $P(\Phi)$:*

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi) = \lim_{n\to\infty} \frac{1}{n} \sum_{S\subset(0,n]} e^{-U_\Phi(S)} \tag{2.4}$$

exists and is infinitely differentiable with respect to Φ on any finite dimensional subspace of $\mathscr{B}$. I.e., *if $\Phi, \psi_1, \ldots, \psi_n \in \mathscr{B}$, then the function $P(\Phi+z_1\psi_1+\cdots+z_n\psi_n)$ has partial derivatives of all orders with respect to $z_1, \ldots, z_n$.*

We remark that since the correlation functions are derivatives of P along directions ψ corresponding to finite range potentials (which therefore belong to $\mathscr{B}$) [9], the above result on the infinite differentiability holds also for the correlation functions.

The proof of this theorem will be based on a theorem by RUELLE [2] (see Theorem 2 below) which will be quoted for further reference after we introduce some definitions.

Let us denote by K_+ the space of the configurations of a semi-infinite system contained in $(0, +\infty)$, *i.e.* the class of lattice subsets $X\subset(0, +\infty)$. It will be useful to consider K_+ as a compact space by introducing on it the topology induced by the following definition of convergence: a sequence $X_n\in K_+$ tends to $X\in K_+$ as $n\to\infty$ if, for every finite interval $(0, l)$, there exists an n_l such that

$$X_n \cap (0, l) \equiv X \cap (0, l), \qquad n > n_l.$$

Now, given two disjoint configurations X, Y such that X is finite and $X\cap Y=\emptyset$, we define the mutual interaction energy $I_\Phi(X|Y)$ between X and Y and the quantity $U_\Phi(X|Y)$ as

$$I_\Phi(X\,|\,Y) = \sum_{\substack{S\cap X\neq\emptyset\\ S\cap Y\neq\emptyset\\ S\subset X\cup Y}} \Phi(S), \tag{2.5}$$

$$U_\Phi(X\,|\,Y) = U_\Phi(X) + I_\Phi(X\,|\,Y). \tag{2.6}$$

Using (2.3), we observe that I_Φ is well defined even if Y is infinite. Finally let us define an operator $\mathscr{L}_\Phi$ acting on the space $C(K_+)$ of the continuous functions on the compact space K_+ (configuration space)

$$(\mathscr{L}_\Phi f)(Y) = \sum_{X \subset (0,1]} e^{-U_\Phi(X|\tau_1 Y)} f(X \cup \tau_1 Y), \qquad f \in C(K_+) \tag{2.7}$$

where $Y \in K_+$ and where X denotes a configuration contained in the lattice points in $(0, 1]$ (*i.e.* either the point 1 or the empty set) and $\tau_1 Y$ is the set $Y+1$ obtained by shifting Y one unit to the right.

If we regard $C(K_+)$ as a Banach space under the sup-norm

$$\|f\| = \sup_{X \in K_+} |f(X)|,$$

then the operator $\mathscr{L}_\Phi$ is a bounded operator on $C(K_+)$, and it has an adjoint $\mathscr{L}_\Phi^*$ defined on the dual space $C^*(K_+)$, *i.e.* on the space of all finite measures on K_+.

In terms of the above definitions, the Ruelle theorem, which essentially gives the main results of the theory of the largest eigenvalue of $\mathscr{L}_\Phi$ and of the associated eigenfunctions, can be stated as follows [2, 4]:

Theorem 2 (Ruelle). *If $\Phi \in \mathscr{B}$, then there exists a number λ_Φ, a function $h_\Phi \in C(K_+)$, a measure $\nu_\Phi \in C^*(K_+)$ such that $\lambda_\Phi > 1$, $h_\Phi > 0$, $\nu_\Phi > 0$, $\nu_\Phi(h_\Phi) = \nu_\Phi(1) = 1$, and furthermore:*

i) *If $Z_n(\Phi)$ is the partition function associated with an interval of length n, we have*

$$e^{-\|\Phi\|} \leqq \lambda_\Phi^{-n} Z_n(\Phi) \leqq e^{\|\Phi\|}$$

where $\|\Phi\|$ has been introduced in (2.3).

ii) $P(\Phi) = \log \lambda_\Phi$, $1 \leqq \lambda_\Phi \leqq 1 + e^{\|\Phi\|}$,

iii) $\mathscr{L}_\Phi h_\Phi = \lambda_\Phi h_\Phi$, $\mathscr{L}_\Phi^* \nu_\Phi = \lambda_\Phi \nu_\Phi$,

iv) $\lim_{n\to\infty} \|(\lambda_\Phi^{-1} \mathscr{L}_\Phi)^n f - \nu_\Phi(f) h_\Phi\| = 0$,

v) *If $\bar{X} = \{\bar{x}_1, \bar{x}_2, \ldots\} \in K_+$ and $\chi_{\bar{X},k}$ is the characteristic function of the set of the X's $\in K_+$ such that $x_i = \bar{x}_i$ $i = 1, 2, \ldots, k$, then*

$$\nu_\Phi(\chi_{\bar{X},k}) > e^{-k\|\Phi\|}(1 + e^{\|\Phi\|})^{-k}.$$

vi) *Let C_m be the subspace of the functions $f \in C(K_+)$ which depend on $X \in K_+$ only through $X \cap (0, m]$,* i.e., *f depends only on the coordinates of X which have values in the interval $(0, m]$; then if $f \in C_m$ and $\nu_\Phi(f) = 0$*

$$\nu_\Phi(|\lambda_\Phi^{-1} \mathscr{L}_\Phi f|) < (1 - e^{-2\|\Phi\|}) \nu_\Phi(|f|).$$

vii) *If $\Phi, \psi \in \mathscr{B}$, then λ_Φ is differentiable, and if $L_\Phi = (\mathscr{L}_\Phi - \lambda_\Phi)$, we have*

$$\nu_\Phi\left(\frac{\partial L_\Phi}{\partial \psi} h_\Phi\right) = 0, \quad \text{i.e.} \quad \frac{\partial \lambda_\Phi}{\partial \psi} = \nu_\Phi\left(\frac{\partial \mathscr{L}_\Phi}{\partial \psi} h_\Phi\right), \tag{2.8}$$

where

$$\frac{\partial \lambda_\Phi}{\partial \psi} = \left.\frac{\partial \lambda_{\Phi + z\psi}}{\partial z}\right|_{z=0} \quad \text{and} \quad \frac{\partial \mathscr{L}_\Phi}{\partial \psi} = \left.\frac{\partial \mathscr{L}_{\Phi + z\psi}}{\partial z}\right|_{z=0},$$

or more explicitly

$$\left(\frac{\partial \mathscr{L}_\Phi}{\partial \psi} f\right)(Y) = - \sum_{X \subset (0,1]} U_\psi(X \mid \tau_1 Y)\, e^{-U_\Phi(X|\tau_1 Y)} f(X \cup \tau_1 Y). \tag{2.9}$$

viii) *h_Φ is norm continuous on Φ when Φ varies in the space $\mathscr{B}$ regarded as a metric space under the distance $\|\Phi - \Phi'\|$, where $\|\cdot\|$ is defined by* (2.3).

RUELLE has in fact shown that the above results are valid for a much wider class of potentials Φ: in fact Theorem 2 holds if we replace the space $\mathscr{B}$ of rapidly decreasing potentials with the space $\bar{\mathscr{B}}$ defined as the completion of $\mathscr{B}$ under the norm (2.3), *i.e.*, defined as the space of the potentials which have a finite first moment.

In this paper we shall be concerned with obtaining a generalization of (vii) by showing the existence of the higher order derivatives of λ_Φ (*i.e.* of $P(\Phi)$) and by obtaining expressions of the form (2.8). To obtain these results, we need to understand better the spectral properties of the operator $\mathscr{L}_\Phi$, and we shall see that RUELLE's analysis of the largest eigenvalue and the relative eigenfunction must be complemented by a careful study of the singular operator $(\mathscr{L}_\Phi - \lambda_\Phi)^{-1}$, *i.e.*, by the analysis of the spectrum of $\mathscr{L}_\Phi$ near the point λ_Φ; we shall be able to perform this analysis only in the very restrictive case $\Phi \in \mathscr{B}$.

In the next section we present our main results concerning the operator $L_\Phi = (\mathscr{L}_\Phi - \lambda_\Phi)$. The properties of L_Φ will be used to prove the infinite differentiability of λ_Φ (Section 4).

§ 3. Some Properties of $L_\Phi^{-1}(\mathscr{L}_\Phi - \lambda_\Phi)^{-1}$

As explained at the end of the preceding section, the analysis of the operator $(\mathscr{L}_\Phi - \lambda_\Phi)^{-1}$ is a prerequisite for the proof of our results.

We shall try to invert the operator $\mathscr{L}_\Phi - \lambda_\Phi$ on a dense subspace of the "orthocomplement" of the eigenvector h_Φ: $\mathscr{E}_\Phi = \{f \in C(K_+);\ \nu_\Phi(f) = 0\}$. A dense subspace on which $L_\Phi = (\mathscr{L}_\Phi - \lambda_\Phi)$ is invertible is the space $\tilde{\mathscr{E}}_\Phi \subset \mathscr{E}_\Phi$ of the functions $f \in \mathscr{E}_\Phi$ such that for some $r_f, t_f > 0$,

$$|f(X) - f(X \cap (0, n])| < r_f \exp(-n^{t_f}). \tag{3.1}$$

We remark for later use that as a consequence of the uniqueness of the eigenfunction h_Φ associated with the eigenvalue λ_Φ (see (iv), Theorem 2), the operator L_Φ is a one-to-one mapping of $\mathscr{E}_\Phi$ into itself and also that any $F \in C(K_+)$ can be uniquely written as $F = f + \alpha h_\Phi$ with $f \in \mathscr{E}_\Phi$ and α real.

Let us now state the main theorem on the operator L_Φ^{-1}.

Theorem 3. *Let $f \in \tilde{\mathscr{E}}_\Phi$. Then there exists a unique vector $L_\Phi^{-1} f \in \tilde{\mathscr{E}}_\Phi$ such that $L_\Phi(L_\Phi^{-1} f) = f$ and*

a) *the vector $L_\Phi^{-1} f$ is given by the absolutely convergent series*

$$L_\Phi^{-1} f = -\lambda_\Phi^{-1} \sum_{k=0}^{\infty} \lambda_\Phi^{-k} \mathscr{L}_\Phi^k f \tag{3.2}$$

and the dependence of $L_\Phi^{-1} f$ has certain continuity properties in Φ and f which are noted in items b), c) *below.*

b) *(Continuity in z of $L_\Phi^{-1} g_z$): Consider a family of functions $g_z \in \mathscr{E}_\Phi$ where z is a parameter varying in a finite dimensional sphere. If g_z is norm-continuous in z and if the numbers r_z, t_z associated with $g_z \in \mathscr{E}_\Phi$ (see* (3.1)) *can be chosen independent of z, then the function $L_\Phi^{-1} g_z \in \mathscr{E}_\Phi$ is norm continuous in z, and we can find two numbers r, $t > 0$ which (according to the definition* (3.1)*) can be associated with both g_z, $L_\Phi^{-1} g_z$ for all z.*

c) *(Continuity in z of $L_{\Phi(z)}^{-1} g_z$): Consider $\psi_1, \ldots, \psi_n \in \mathscr{B}$ and the operator $L_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}$ and suppose that $\{z_1, \ldots, z_n\}$ vary in an n-dimensional sphere; suppose also that we are given a family of functions $g_z \in \mathscr{E}_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}$ such that the parameters r, t associated with g_z (see* (3.1)*) can be chosen independent of z. Then*

$$L_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}^{-1} g_z$$

is a norm continuous function of z, and the parameters r, t associated with g_z can be chosen to be good also for $L_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}^{-1} g_z$.

The proof of items b), c) are implicitly obtained in proving a). A sketch of the proof of a) is given in the Appendix.

In order to apply the preceding theorem, we shall need the lemma below which will enable us to prove that the functions f which will be encountered in the next section are in $\mathscr{E}_\Phi$.

Lemma 1. a) *The function h_Φ has the property*

$$|h_\Phi(X) - h_\Phi(X \cap (0, n])| < r \exp(n^{-t}) \tag{3.3}$$

for some r, $t > 0$ and for $\Phi \in \mathscr{B}$; r, t can be chosen to be Φ-independent if Φ varies in a bounded finite dimensional subset of $\mathscr{B}$.

b) *Let $\psi_1, \ldots, \psi_n \in \mathscr{B}$, and let us consider the operators defined by*

$$\frac{\partial^n \mathscr{L}_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}}{\partial\psi_1 \ldots \partial\psi_n} = \frac{\partial^n \mathscr{L}_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}}{\partial z_1 \ldots \partial z_n}. \tag{3.4}$$

If we apply this operator to any function f which verifies (3.1), *the function*

$$\frac{\partial^n \mathscr{L}_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}}{\partial\psi_1 \ldots \partial\psi_n} f$$

still has the property (3.1) *for all $\{z_1, \ldots, z_n\}$ and the numbers r and t can be chosen independent of $\{z_1, \ldots, z_n\}$ if $\{z_1, \ldots, z_n\}$ vary in a bounded set.*

The proof of this lemma can be obtained by combining item iv) (with $f = 1$) and item i) of Theorem 2. We omit it.

Remark. The operators in (3.4) are explicitly given by

$$\begin{aligned}\left[\frac{\partial^n \mathscr{L}_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}}{\partial\psi_1 \ldots \partial\psi_n} f\right](Y) = (-1)^n \sum_{X \subset (0,1]} U_{\psi_1}(X \mid \tau_1 Y) \ldots U_{\psi_n}(X \mid \tau_1 Y) \\ \cdot \exp(-U_{\Phi + z_1\psi_1 + \cdots + z_n\psi_n}(X \mid \tau_1 Y)) \cdot f(X \cup \tau_1 Y).\end{aligned} \tag{3.5}$$

§ 4. The Derivatives of λ_Φ

The existence of the first derivative of λ_Φ has already been obtained in (vii), Theorem 2. In this section we show that this formula can be generalized as

$$\nu_\Phi\left(\sum_{k=1}^{n}\sum_{i_1,\ldots,i_k}\frac{\partial^k L_\Phi}{\partial\psi_{i_1}\ldots\partial\psi_{i_k}}\frac{\partial^{n-k}h_\Phi}{\partial\psi_{i'_1}\ldots\partial\psi_{i'_{n-k}}}\right)=0, \tag{4.1}$$

where the sum runs over all the sets of k indices chosen from $\{1,2,\ldots,n\}$ and $\{i'_1,\ldots,i'_{n-k}\}$ denotes the complement of $\{i_1,\ldots,i_k\}$ in $\{1,2,\ldots,n\}$ and

$$\frac{\partial^{n-k}h_\Phi}{\partial\psi_{i'_1},\ldots,\partial\psi_{i'_{n-k}}}$$

are functions defined below.

A heuristic argument for (4.1) is based on the identity $L_\Phi h_\Phi=0$, which, if differentiated, becomes

$$\sum_{k=0}^{n}\sum_{i_1,\ldots,i_k}\frac{\partial^k L_\Phi}{\partial\psi_{i_1}\ldots\partial\psi_{i_k}}\frac{\partial^{n-k}h_\Phi}{\partial\psi_{i'_1}\ldots\partial\psi_{i'_{n-k}}}=0, \tag{4.1'}$$

and integrating both sides of this identity with respect to ν_Φ, we get (4.1), and the term with $k=0$ vanishes because of (iii), Theorem 2.

We shall in fact see that the argument is really formal because we are unable to prove that h is differentiable. Nevertheless we shall show that the above argument can be made rigorous by conveniently defining the function $\partial^k h_\Phi/\partial\psi_1\ldots\partial\psi_k$ which will turn out to be transverse derivatives of h_Φ with respect to the subspace $\mathscr{E}_\Phi$.

Let us first give a precise meaning to the quantities in (4.1); for this purpose we need only to define the symbols $\partial^k\lambda_\Phi/\partial\psi_1\ldots\partial\psi_k$, $\partial^k h_\Phi/\partial\psi_1\ldots\partial\psi_k$ (the symbols $\partial^k\mathscr{L}_\Phi/\partial\psi_1,\ldots,\partial\psi_k$ have already been defined in (3.4), (3.5)). We give the following inductive definition: suppose that

$$\frac{\partial\lambda_\Phi}{\partial\psi_1},\ldots,\frac{\partial^n\lambda_\Phi}{\partial\psi_1\ldots\partial\psi_n};\quad \frac{\partial h_\Phi}{\partial\psi_1},\ldots,\frac{\partial^{n-1}h_\Phi}{\partial\psi_1\ldots\partial\psi_{n-1}} \tag{4.2}$$

have already been defined for all $\psi_1,\ldots,\psi_n\in\mathscr{B}$, and suppose that $\partial^k\lambda_\Phi/\partial\ldots$ are real numbers and

$$\frac{\partial^k h_\Phi}{\partial\psi_1\ldots\partial\psi_k}\in\tilde{\mathscr{E}}_\Phi,\qquad n-1\geqq k\geqq 1, \tag{4.3}$$

$$\nu_\Phi\left(\sum_{k=1}^{m}\sum_{i_1\ldots i_k}\frac{\partial^k(\mathscr{L}_\Phi-\lambda_\Phi)}{\partial\psi_{i_1}\ldots\partial\psi_{i_k}}\frac{\partial^{m-k}h_\Phi}{\partial\psi_{i'_1}\ldots\partial\psi_{i'_{m-k}}}\right)=0 \tag{4.4}$$

for $m<n$. It follows, therefore, that the function integrated in (4.4) with respect to ν_Φ is in $\mathscr{E}_\Phi$ (we are using item b) of Lemma 1 in the preceding section and (4.3)); applying a) of Theorem 3, we can now define

$$\frac{\partial^n h_\Phi}{\partial\psi_1\ldots\partial\psi_n}=-L_\Phi^{-1}\left(\sum_{k=1}^{n}\sum_{i_1\ldots i_k}\frac{\partial^k(\mathscr{L}_\Phi-\lambda_\Phi)}{\partial\psi_{i_1}\ldots\partial\psi_{i_k}}\frac{\partial^{n-k}h_\Phi}{\partial\psi_{i'_1}\ldots\partial\psi_{i'_{n-k}}}\right) \tag{4.5}$$

(this formula is heuristically suggested by (4.1′)), and we can also define $\partial^{n+1}\lambda_\Phi/\partial\psi_1\ldots\partial\psi_{n+1}$ (using (4.5)) in such a way that (4.1) holds with n replaced by $n+1$. So we see that in order to define inductively the symbols (4.2) we need only to define $\partial^1\lambda_\Phi/\partial\psi_1$ in such a way that (4.1) holds for $n=1$, *i.e.*, in such a way that

$$\nu_\Phi\left(\left(\frac{\partial\mathscr{L}_\Phi}{\partial\psi_1}-\frac{\partial^1\lambda_\Phi}{\partial\psi_1}\right)h_\Phi\right)=0 \tag{4.6}$$

and

$$\left(\frac{\partial\mathscr{L}_\Phi}{\partial\psi_1}-\frac{\partial^1\lambda_\Phi}{\partial\psi_1}\right)h_\Phi\in\tilde{\mathscr{E}}_\Phi, \tag{4.7}$$

and this can be done by defining $\partial^1\lambda_\Phi/\partial\psi_1$ to be the derivative of λ_Φ in the direction ψ_1:

$$\frac{\partial^1\lambda_\Phi}{\partial\psi_1}=\left.\frac{\partial\lambda_{\Phi+z\psi_1}}{\partial z}\right|_{z=0} \tag{4.8}$$

and by using vii), Theorem 2, to insure (4.6) and Lemma 1 to insure (4.7).

The remaining part of this section is devoted to proof that the symbols $\partial^n\lambda_\Phi/\partial\psi_1\ldots\partial\psi_n$ are the derivatives of λ_Φ, *i.e.*,

$$\frac{\partial^n\lambda_\Phi}{\partial\psi_1\ldots\partial\psi_n}=\left.\frac{\partial^n\lambda_{\Phi+z_1\psi_1+\cdots+z_n\psi_n}}{\partial z_1\ldots\partial z_n}\right|_{z_1=z_2=\cdots=z_n=0}. \tag{4.9}$$

Since a clear idea of this proof can be gotten from the cases $n=2$, 3, we treat only these two cases, and for the general case we shall only sketch the induction argument.

Consider the case of the second derivative: if $\phi,\psi\in\mathscr{B}$, then using the formulae

$$\frac{\partial\lambda_\Phi}{\partial\psi}=\nu_\Phi\left(\frac{\partial\mathscr{L}_\Phi}{\partial\psi}h_\Phi\right)\quad\text{and}\quad \nu_\Phi(L_\Phi f)=0\qquad\text{for all } f\in C(K_+)$$

and items b), c) of Theorem 3, we can go through the following chain of equations:

$$\begin{aligned}
&\frac{1}{z}\left(\frac{\partial\lambda_{\Phi+z\phi}}{\partial\psi}-\frac{\partial\lambda_\Phi}{\partial\psi}\right)=\\
&=\frac{\nu_{\Phi+z\phi}-\nu_\Phi}{z}\left(\frac{\partial\mathscr{L}_{\Phi+z\phi}}{\partial\psi}h_{\Phi+z\phi}\right)+\\
&\quad+\frac{1}{z}\nu_\Phi\left(\frac{\partial\mathscr{L}_{\Phi+z\phi}}{\partial\psi}-\frac{\partial\mathscr{L}_\Phi}{\partial\psi}\right)h_{\Phi+z\phi}+\nu_\Phi\left(\frac{\partial\mathscr{L}_\Phi}{\partial\psi}\,\frac{h_{\Phi+z\phi}-h_\Phi}{z}\right)\\
&=-\frac{1}{z}\nu_\Phi\left(\frac{\partial L_{\Phi+z\phi}}{\partial\psi}h_{\Phi+z\phi}\right)+\frac{1}{z}\,\frac{\partial\lambda_{\Phi+z\phi}}{\partial\psi}(\nu_{\Phi+z\phi}-\nu_\Phi)\,h_{\Phi+z\phi}\\
&\quad+\frac{1}{z}\nu_\Phi\left(\left(\frac{\partial\mathscr{L}_{\Phi+z\phi}}{\partial\psi}-\frac{\partial\mathscr{L}_\Phi}{\partial\psi}\right)h_{\Phi+z\phi}\right)+\nu_\Phi\left(\frac{\partial L_\Phi}{\partial\psi}\,\frac{h_{\Phi+z\phi}-h_\Phi}{z}\right)\\
&\quad+\frac{\partial\lambda_\Phi}{\partial\psi}\nu_\Phi\left(\frac{h_{\Phi+z\phi}-h_\Phi}{z}\right)
\end{aligned}$$

and, using (4.5) in the form $L_\Phi \dfrac{\partial h_\Phi}{\partial\psi} \equiv -\dfrac{\partial\lambda_\Phi}{\partial\psi} h_\Phi$, we have

$$= v_\Phi\left(\frac{L_{\Phi+z\phi}}{z}\,\frac{\partial h_{\Phi+z\phi}}{\partial\psi}\right) + \frac{1}{z} v_\Phi\left(\frac{\partial\mathscr{L}_{\Phi+z\phi}}{\partial\psi} - \frac{\partial\mathscr{L}_\Phi}{\partial\psi}\right) h_{\Phi+z\phi}$$

$$+ v_\Phi\left(\frac{\partial L_\Phi}{\partial\psi}\,\frac{h_{\Phi+z\phi}-h_\Phi}{z}\right) + \frac{1}{z}\left(\frac{\partial\lambda_{\Phi+z\phi}}{\partial\psi} - \frac{\partial\lambda_\Phi}{\partial\psi}\right)\cdot v_\Phi(h_\Phi - h_{\Phi+z\phi}).$$

Hence if $z\to 0$ (taking into account viii) of Theorem 2), we have

$$\lim_{z\to 0}\frac{1}{z}\left(\frac{\partial\lambda_{\Phi+z\phi}}{\partial\psi} - \frac{\partial\lambda_\Phi}{\partial\psi}\right) = v_\Phi\left(\frac{\partial L_\Phi}{\partial\phi}\,\frac{\partial h_\Phi}{\partial\psi}\right) + v_\Phi\left(\frac{\partial^2\mathscr{L}_\Phi}{\partial\psi\,\partial\phi} h_\Phi\right)$$
$$+ \lim_{z\to 0} v_\Phi\left(\frac{\partial L_\Phi}{\partial\psi}\cdot\frac{h_{\Phi+z\phi}-h_\Phi}{z}\right). \tag{4.10}$$

Now let us introduce the projection operator E_Φ from $C(K_+)$ on $\mathscr{E}_\Phi$: $E_\Phi f = f - v_\Phi(f)\, h_\Phi$, and consider $E_\Phi \dfrac{1}{z}(h_{\Phi+z\phi} - h_\Phi)$. Using the identity $L_\Phi h_\Phi = 0$, we find that

$$\frac{L_{\Phi+z\phi} - L_\Phi}{z} h_{\Phi+z\phi} = -L_\Phi \frac{h_{\Phi+z\phi} - h_\Phi}{z} = -L_\Phi E_\Phi \frac{h_{\Phi+z\phi} - h_\Phi}{z}. \tag{4.11}$$

Clearly, it follows from (4.11) that $\dfrac{1}{z}(L_{\Phi+z\phi} - L_\Phi)\, h_{\Phi+z\phi} \in \mathscr{E}_\Phi$ and as a function of z, fulfills the hypothesis of Lemma 1. Therefore, from (4.11) (4.5) we get

$$\lim_{z\to\infty} E_\Phi \frac{h_{\Phi+z\phi} - h_\Phi}{z} = -L_\Phi^{-1}\frac{\partial L_\Phi}{\partial\phi} h_\Phi = \frac{\partial h_\Phi}{\partial\phi}. \tag{4.12}$$

Using (4.10) (4.11), we see that we have proved that

$$\lim_{z\to 0}\frac{1}{z}\left(\frac{\partial\lambda_{\Phi+z\phi}}{\partial\psi} - \frac{\partial\lambda_\Phi}{\partial\psi}\right) = v_\Phi\left(\frac{\partial L_\Phi}{\partial\psi}\,\frac{\partial h_\Phi}{\partial\phi}\right) + v_\Phi\left(\frac{\partial^2 L_\Phi}{\partial\psi\,\partial\phi} h_\Phi\right)$$
$$+ v_\Phi\left(\frac{\partial L_\Phi}{\partial\phi}\,\frac{\partial h_\Phi}{\partial\psi}\right) \tag{4.13}$$

and that the right hand side in (4.13) is also the definition of the symbol $\partial^2\lambda_\Phi/\partial\psi\,\partial\phi$. Therefore $\partial^2\lambda_\Phi/\partial\psi\,\partial\phi$ coincides with the second derivative of λ_Φ whose existence is proved by (4.13). The continuity of $\partial^2\lambda_\Phi/\partial\psi\,\partial\phi$ on the finite dimensional subspaces of $\mathscr{B}$ follows from the continuity of the first derivatives of L_Φ and continuity of $\partial h_\Phi/\partial\psi$, $\partial h_\Phi/\partial\phi$ and v_Φ.

At this point one could naively guess that the general formula for $\partial^n h_\Phi/\partial\psi_1, \ldots, \partial\psi_n$ is simply

$$\frac{\partial^n h_\Phi}{\partial\psi_1\,\partial\psi_2, \ldots, \partial\psi_n} = \lim_{z\to 0} E_\Phi \frac{1}{z}\left(\frac{\partial^{n-1} h_{\Phi+z\psi_n}}{\partial\psi_1, \ldots, \partial\psi_{n-1}} - \frac{\partial^{n-1} h_\Phi}{\partial\psi_1, \ldots, \partial\psi_{n-1}}\right).$$

Since this is not the case, we give the derivation of the formula for the third derivative from which the general situation can be easily inferred.

Let $\phi, \psi, \chi \in \mathscr{B}$ be three potentials, and define (as from (4.5))

$$\frac{\partial^2 h_\Phi}{\partial \phi \, \partial \psi} = -L_\Phi^{-1} \left(\frac{\partial L_\Phi}{\partial \psi} \frac{\partial h_\Phi}{\partial \phi} + \frac{\partial^2 L_\Phi}{\partial \psi \, \partial \phi} h_\Phi + \frac{\partial L_\Phi}{\partial \phi} \frac{\partial L_\Phi}{\partial \psi} \right),$$

and similarly define $\partial^2 h_\Phi / \partial \phi \, \partial \chi$, $\partial^2 h_\Phi / \partial \psi \, \partial \chi$. Let $h_{\Phi+z\phi} - h_\Phi = E_\Phi (h_{\Phi+z\phi} - h_\Phi) + \alpha_z(\phi) \, h_\Phi$. Now to prove an equation of the type of (4.12), we consider the identity which follows from the definition (4.5) for $n=1$:

$$L_\Phi \frac{\partial h_\Phi}{\partial \psi} + \frac{\partial L_\Phi}{\partial \psi} h_\Phi = 0 .$$

From this we deduce

$$\begin{aligned} &\frac{L_{\Phi+z\phi} - L_\Phi}{z} \frac{\partial h_{\Phi+z\phi}}{\partial \psi} + L_\Phi \frac{1}{z} \left(\frac{\partial h_{\Phi+z\phi}}{\partial \psi} - \frac{\partial h_\Phi}{\partial \psi} \right) \\ &\quad + \frac{1}{z} \left(\frac{\partial L_{\Phi+z\phi}}{\partial \psi} - \frac{\partial L_\Phi}{\partial \psi} \right) h_{\Phi+z\phi} + \frac{\partial L_\Phi}{\partial \psi} E_\Phi \frac{h_{\Phi+z\phi} - h_\Phi}{z} + \frac{\alpha_z(\phi)}{z} \frac{\partial L_\Phi}{\partial \psi} h_\Phi = 0, \end{aligned}$$

and then it follows that

$$\begin{aligned} &E_\Phi \frac{1}{z} \left(\frac{\partial h_{\Phi+z\phi}}{\partial \psi} - \frac{\partial h_\Phi}{\partial \psi} \right) + \frac{\alpha_z(\phi)}{z} L_\Phi^{-1} \frac{\partial L_\Phi}{\partial \psi} h_\Phi \\ &= -L_\Phi^{-1} \frac{L_{\Phi+z\phi} - L_\Phi}{z} \frac{\partial h_{\Phi+z\phi}}{\partial \psi} + \frac{1}{z} \left(\frac{\partial L_{\Phi+z\phi}}{\partial \psi} - \frac{\partial L_\Phi}{\partial \psi} \right) h_{\Phi+z\phi} + \frac{\partial L_\Phi}{\partial \psi} E_\Phi \frac{h_{\Phi+z\phi} - h_\Phi}{z} . \end{aligned}$$

Using item b), c) of Theorem 3 and Lemma 1 and (4.5), (4.12), we find that

$$\frac{\partial^2 h_\Phi}{\partial \phi \, \partial \psi} = \lim_{z \to 0} E_\Phi \frac{1}{z} \left(\frac{\partial h_{\Phi+z\phi}}{\partial \psi} - \frac{\partial h_\Phi}{\partial \psi} \right) + \frac{\alpha_z(\phi)}{z} L_\Phi^{-1} \frac{\partial L_\Phi}{\partial \psi} h_\Phi . \tag{4.14}$$

Now we can easily compute the third derivative of λ_Φ, and, after a straightforward but lengthy and cumbersome calculation, we get

$$\begin{aligned} \lim_{z \to 0} \frac{1}{z} \left(\frac{\partial^2 \lambda_{\Phi+z\chi}}{\partial \phi \, \partial \psi} - \frac{\partial^2 \lambda_\Phi}{\partial \phi \, \partial \psi} \right) &= \nu_\Phi \left(\frac{\partial^2 L_\Phi}{\partial \phi \, \partial \psi} \frac{\partial h_\Phi}{\partial \chi} \right) + \nu_\Phi \left(\frac{\partial^2 L_\Phi}{\partial \phi \, \partial \chi} \frac{\partial h_\Phi}{\partial \psi} \right) \\ &\quad + \nu_\Phi \left(\frac{\partial^2 L_\Phi}{\partial \psi \, \partial \chi} \frac{\partial h_\Phi}{\partial \phi} \right) + \nu_\Phi \left(\frac{\partial L_\Phi}{\partial \chi} \frac{\partial^2 h_\Phi}{\partial \phi \, \partial \psi} \right) + \nu_\Phi \left(\frac{\partial L_\Phi}{\partial \psi} \frac{\partial^2 h_\Phi}{\partial \chi \, \partial \psi} \right) \\ &\quad + \nu_\Phi \left(\frac{\partial L_\Phi}{\partial \phi} \frac{\partial^2 h_\Phi}{\partial \chi \, \partial \psi} \right) + \nu_\Phi \left(\frac{\partial^2 L_\Phi}{\partial \phi \, \partial \psi \, \partial \chi} h_\Phi \right) . \end{aligned} \tag{4.15}$$

Since the right hand side of (4.15) coincides with the definition of the symbol $\partial^3 \lambda_\Phi / \partial \phi \, \partial \psi \, \partial \chi$, we have concluded the proof of the existence of the third derivative of λ_Φ and that it coincides with the symbol $\partial^3 \lambda_\Phi / \partial \phi \, \partial \psi \, \partial \chi$.

It is now obvious how to generalize, by induction, the preceding argument to the n^{th} derivative. We give only the formula for $\partial^n h_\Phi / \partial \psi_1 \dots \partial \psi_n$, which generalizes

(4.14). Let us write

$$\begin{aligned}\frac{\partial^k h_{\Phi+z\psi n}}{\partial\psi_{i_1}\dots\partial\psi_{i_k}}-\frac{\partial^k h_\Phi}{\partial\psi_{i_1}\dots\partial\psi_{i_k}}\\ =E_\Phi\left(\frac{\partial^k h_{\Phi+z\psi n}}{\partial\psi_{i_1}\dots\partial\psi_{i_k}}-\frac{\partial^k h_\Phi}{\partial\psi_{i_1}\dots\partial\psi_{i_k}}\right)+\alpha_z(\psi_{i_1},\dots,\psi_{i_k},\psi_n)\,h_\Phi\end{aligned}\tag{4.16'}$$

for $k\leqq n-1$, $i_1,\dots,i_k\leqq n-1$. Then it can be proved, by an argument similar to that of (4.14), that

$$\begin{aligned}\lim_{z\to 0}E_\Phi\frac{1}{z}\left(\frac{\partial^{n-1} h_{\Phi+z\psi n}}{\partial\psi_1\dots\partial\psi_{n-1}}-\frac{\partial^{n-1} h_\Phi}{\partial\psi_1\dots\partial\psi_{n-1}}\right)-\sum_{k=0}^{n-2}\sum_{i_1,\dots,i_k\leqq n-1}\\ \cdot\frac{\alpha_z(\psi_{i_1},\dots,\psi_{i_k},\psi_n)}{z}\frac{\partial^{n-1-k}h_\Phi}{\partial\psi_{i'_1},\dots,\partial\psi_{i'_{n-1-k}}}=\frac{\partial^n h_\Phi}{\partial\psi_1\dots\partial\psi_n}\end{aligned}\tag{4.16''}$$

where $\partial^n h_\Phi/\partial\psi_i\dots\partial\psi_n$ is defined in (4.5). Now from (4.16) one can prove the existence of the $(n+1)^{\text{th}}$ derivative of λ_Φ and its coincidence with the number $\partial^{n+1}\lambda_\Phi/\partial\psi_1\dots\partial\psi_{n+1}$.

Appendix

We use the symbols introduced in § 3 throughout the Appendix; however, we shall not write the subscript Φ (*e.g.*, in L_Φ, h_Φ, *etc.*).

Theorem 3 is a consequence of the following lemmas.

Lemma A.1. *If* $g\in\mathscr{E}$, *then there exist* $r',t'>0$ *such that*

$$\nu(|(\lambda^{-1}\mathscr{L})^n g|)<r'\exp(-n^{t'}),\qquad n>0.\tag{A.1}$$

The proof is based on (2.2), (3.1) and item i) of Theorem 2 which lead to the inequality

$$|(\lambda^{-1}\mathscr{L})^m g(Y)-(\lambda^{-1}\mathscr{L})^m g(Y\cap(0,n])|<r_0\exp(-n^{t_0}),\tag{A.2}$$

where r_0, t_0 are some positive numbers (determined by g). Then, using this inequality and the techniques, due to RUELLE [2], which were used to prove lemmas 6, 7 in [4], one obtains (A.1) with $r'>r_0$, $t'<t_0$.

Lemma A.2. *If* $g\in\mathscr{E}$, *then there exist* $r'',t''>0$ *such that*

$$\|\lambda^{-n}\mathscr{L}^n g\|<r''\exp(-n^{t''}).\tag{A.3}$$

Proof. Let $Y_n\in K_+$ *maximize* $|\lambda^{-n}\mathscr{L}^n g|$, *i.e.*, $|\lambda^{-n}\mathscr{L}^n g(Y_n)|=\|\lambda^{-n}\mathscr{L}^n g\|$, and define

$$\Gamma_n=\{X\in K_+:\|\lambda^{-n}\mathscr{L}^n g\|-|\lambda^{-n}\mathscr{L}^n g(X)|<2r'\exp(-\psi(n)^{t'})\}\tag{A.4}$$

where r', t' are the same as in Lemma A.1 and $\psi(n)$ is an integer valued non-decreasing function to be determined later. From formula (A.2) it follows that if $X\cap(0,\psi(n)]=Y_n\cap(0,\psi(n)]$, then

$$|\lambda^{-n}\mathscr{L}^n g(X)-\lambda^{-n}\mathscr{L}^n g(Y_n)|<2r_0\exp(-\psi^{t_0}(n))<2r'\exp(-\psi^{t'}(n)).\tag{A.5}$$

Now from (A.4) and (A.5) it follows that the set Γ_n contains the set $S_n = \{X \in K_+: X \cap (0, \psi(n)] = Y_n \cap (0, \psi(n)]\}$, and from (v) of Theorem 2 it follows that

$$\nu(\Gamma_n) \geqq \nu(S_n) \geqq e^{-\psi(n)\,\|\Phi\|}(1+e^{\|\Phi\|})^{-\psi(n)}, \tag{A.6}$$

so we find from (A.6) that

$$\begin{aligned}\nu(|\lambda^{-n}\mathscr{L}^n g|) &\geqq \nu(\Gamma_n)(\|\lambda^{-n}\mathscr{L}^n g\| - 2r'\exp(-\psi^{t'}(n)))\\ &\geqq (1+e^{\|\Phi\|})^{-2\psi(n)}(\|\lambda^{-n}\mathscr{L}^n g\| - 2r'\exp(-\psi^{t'}(n))).\end{aligned}$$

Using Lemma A.1, we have

$$\|\lambda^{-n}\mathscr{L}^n g\| < (1+e^{\|\Phi\|})^{2\psi(n)} \cdot r'\exp(-n^{t'}) + 2r'\exp(-\psi^{t'}(n)),$$

and so Lemma A.2 follows by choosing, say, $\psi(n)$ = the largest integer less than $n^{(\frac{1}{2})t'}$.

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After completing this paper, we received a preprint of H. Araki, "Gibbs states of a one dimensional quantum lattice" (now published in Comm. Math. Phys. **14**, 120, 1969). This paper deals with exponentially decreasing interactions and much more general quantum lattice systems, and in it is achieved the proof of analyticity of the thermodynamic pressure in the relevant parameters. Taking into account of results Araki, we can complete McCoy's conjecture as follows: If the parameter t in formula (2.2) is $\geqq 1$, we have analyticity (as already proved by Araki), and if $t<1$, we have infinite differentiability (as proved here) but *not* analyticity.

References

1. McCoy, B. M., Private communication.
2. Ruelle, D., Comm. Math. Phys. **9**, 267 (1968).
3. Gallavotti, G., S. Miracle-Sole, & D. Ruelle, Phys. Lett. **26**A, 350 (1968).
4. Gallavotti, G., & S. Miracle-Sole, J. Math. Phys. **11**, 147 (1970).
5. Fisher, M. E., Arch. Rational Mech. Anal. **17**, 377 (1964). See also [9].
6. Fisher, M. E., Physics **3**, 255 (1967).
7. Dyson, F. J., Comm. Math. Phys. **12**, 91 (1969).
8. Dyson, F. J., Non-Existence of Spontaneous Magnetization in a One-Dimensional Ising Ferromagnet. Institute for Advanced Study, Princeton, N.J., preprint.
9. See for instance M. E. Fisher: J. Math. Phys. **6**, 1643 (1965). See also G. Gallavotti, & S. Miracle-Sole: Comm. Math. Phys. **5**, 317 (1967).

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