THE EXPONENTIAL INTERACTION IN R^n

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We prove that the Schwinger functions for the ultraviolet cut-off exponential interaction with euclidean measure exp $\{-\lambda \int_\Lambda e^{\alpha\xi_k(x)}\, dx\} d\mu_0(\xi) / \int \exp\{-\lambda \int_\Lambda \cdot e^{\alpha\xi_k(x)}\, dx\} d\mu_0(\xi)$, $\lambda > 0$, converge as the ultraviolet cut-off is removed. The limits are the free Schwinger functions in the case of space–time dimension $n \geqslant 3$. In the case $n = 2$ this holds for $|\alpha|$ sufficiently big, whereas for $|\alpha| < 2\sqrt{\pi}$, one has the well-known nontrivial Schwinger functions of the exponential interaction.

The exponential interaction was introduced as a model of relativistic quantum fields in refs. [1,2]. It was pointed out by us in ref. [2] that in the case of space–time dimension $n = 2$ the exponential interaction $m_\Lambda(\xi) = \int_\Lambda : e^{\alpha\xi} \cdot (x)\, d\nu(\alpha)\, dx$ (ξ being the euclidean free field) exists and is a positive function in $L_2(d\mu_0)$, if ν is a bounded positive measure with support in $(-2\sqrt{\pi}, 2\sqrt{\pi})$, μ_0 being the free euclidean field. We also proved for supp $\nu \subset (-4/\sqrt{\pi}, 4/\sqrt{\pi})$ that the corresponding Schwinger functions are decreasing functions of Λ, and that the limit $\Lambda \uparrow R^2$ exists, is nontrivial, and defines euclidean fields leading to relativistic quantum fields satisfying all the usual postulates. It follows also from the estimates in ref. [2] that, for arbitrary n, one has a priori estimates for the Schwinger functions of a space and time and momentum cut-off exponential interaction in terms of the free Schwinger functions. For this reason it is interesting to study the exponential interaction for arbitrary space–time dimension n.

Since our work in ref. [2], closely related work has appeared [3–9] and in particular the range for existence of relativistic models with exponential interactions has been extended to the interval $(-2\sqrt{\pi}, 2\sqrt{\pi})$ [6]. Further discussion of the removal of the cut-offs by subsequences is contained in refs. [3,5–10].

The first proof of the convergence of the ultraviolet cut-off measures, when the cut-off is removed for arbitrary n, was given in ref. [11] and will appear in ref. [13]. There is a new proof of this fact in a recent letter [9].

We show in this paper that for an arbitrary space–time dimension n the measures defining the ultraviolet cut-off exponential interaction converge weakly as the cut-off is removed and the limit then defines the exponential interaction in a finite region of the euclidean space of dimension n. Moreover this limit is obtained as a limit of an a.e. convergent sub martingale, and thus in particular the Schwinger functions are convergent. That is if $m_k(\xi) = \int_\Lambda : e^{\alpha\xi_k(x)} : dx$, where $\xi_k(x)$ is the ultraviolet cut-off free euclidean field, then $e^{-\lambda m_k(\xi)}$ is an a.e. convergent sub martingale in k with respect to μ_0, the probability measure for the free euclidean field. In fact, $m_k(\xi)$ is a positive martingale and we prove that in any dimension n, $m_k(\xi)\, d\mu_0(\xi) \to \int_\Lambda \mu_0(\xi + \alpha G_x)\, dx$, where $G_x(y) = G(x - y)$ is the kernel of $(-\Delta + m^2)^{-1}$, as $k \to \infty$, weakly as measures.

We show moreover that for dimension $n \geqslant 4$ we have that the measures $\mu_0(\xi)$ and the $\int \mu_0(\xi + \alpha G_x)\, dx$ have disjoint supports. This is due to estimates involving in a detailed way the singularity of the covariance for μ_0, i.e. the free propagator. This result together with the sub martingale convergence permits us to prove that $e^{-\lambda m_k(\xi)} d\mu_0(\xi) \to d\mu_0(\xi)$ weakly as $k \to \infty$. This implies in particular that the Schwinger functions of the regularized exponential interaction converge as $k \to \infty$ to the free Schwinger functions, for $n \geqslant 4$. A proof [15] holding also for the cases $n = 3$ and ($n = 2$,

$|\alpha|$ big) uses results from ref. [14]. Hence the exponential interaction, defined as

$$\exp\{-\lambda \int .e^{\alpha\xi(x)}\ d\nu(\alpha)\} d\mu_0(\xi) /$$

$$\exp\{-\lambda \int e^{\alpha\xi(x)\cdot}\ d\nu(\alpha)\} d\nu_0(\xi) ,$$

is trivial for $n \geqslant 3$ and ($n = 2$, $|\alpha|$ big). This answers questions raised in refs. [3,16] and particularly in the recent letter [9] by Ranczka.

As we recalled above, the exponential interaction is nontrivial for $n = 2$ and $\alpha^2 < 4\pi$ (and for $n = 1$ and all α). Whereas the question of existence is solved for all n and all α, the question of nontriviality remains open for $n = 2$ when $4\pi < \alpha^2 < \alpha_0^2$. Our result can be adapted to show that any exponential interaction obtained as a limit of regularized (ultraviolet or lattice cut-off, with different boundary conditions) interactions is trivial, for arbitrary Borel measures ν, as long as one keeps λ finite. The question whether it is possible to obtain nontriviality by allowing more general ν concentrated at 0 remains open.

Finally we want to point out that the triviality of the exponential interaction is related to the problem of the irreducibility of the energy representation of the Sobolev–Lie groups of mappings of R^n into a compact Lie group [12].

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