On the Ultraviolet Stability in Statistical Mechanics and Field Theory (*).

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Summary. – *The ultraviolet stability and its relation to the rarefied polymer theory are analyzed for the hierarchical Markov fields in $d = 1, 2$ space dimensions. We present a complete derivation of an upper bound to the ground state energy and we give a detailed sketch of the procedure for obtaining a lower bound with the same technique employed to derive the upper bound.*

1. – Introduction and notations.

The definition of the basic object of this paper, « the Markov hierarchical field », (1), is given in terms of a sequence $(\bar{Q}_i)_{i=0}^{\infty}$ of compatible pavements of R^{d+1} with cubic (open) tesserae.

The side of a tessera $\Delta \in \bar{Q}_N$ in assumed to be $1_N = 2^{-N}$ and the compatibility of the pavements means that each $\Delta \in \bar{Q}_N$ is exactly paved by the tesserae of $\bar{Q}_{N+1}$ contained in it.

To each $\Delta \in \bigcup_{N=0}^{\infty} \bar{Q}_N$ we associate a gaussian random variable z_Δ. The distribution of the variables $(z_\Delta)_{\Delta \in \bar{Q}_N}$ is described by the formal density:

$$\text{const} \exp -\frac{\beta}{2}\Big[\sum_{(\Delta,\Delta')\in\bar{Q}_N}^{*} (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_{\Delta\in\bar{Q}_N} z_\Delta^2\Big]$$

where $\sum^*$ runs over the pairs of nearest neighbour tesserae of $\bar{Q}_N$.

The variables z_Δ and $z_{\Delta'}$ will be assumed independent if $|\Delta| = (\text{volume of } \Delta) \neq |\Delta'|$.

More precisely the random field $(z_\Delta)_{\Delta \in \bigcup_{N=0}^{\infty} \bar{Q}_N}$ is the gaussian random field with covariance:

$$E(z_\Delta z_{\Delta'}) = \delta_{NM} \bar{C}^{[N]}_{\Delta\Delta'} \qquad \text{if } \Delta \in \bar{Q}_N,\ \Delta' \in \bar{Q}_M$$

where the matrix $\bar{C}^{[N]}$ is defined by:

$$\bar{C}^{[N]}_{\Delta\Delta'} = \frac{\beta^{-1}}{(2\pi)^{d+1}} \int_{-\pi}^{\pi} \frac{\exp ik\cdot(\xi_\Delta - \xi_{\Delta'})\, dk}{\Big(\alpha^2 + 2\sum_{i=1}^{d+1}(1-\cos k_i)\Big)}$$

where $\xi_\Delta =$ centre of Δ.

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The « free hierarchical Markov field » with « ultraviolet cut off of order N or length 2^{-N} » is the gaussian random field on R^{d+1} defined as:

$$\varphi_\xi^{[\leqslant N]} = \sum_{k=0}^{N} \sum_{\Delta \in \bar{Q}_k} \sqrt{2\gamma_k}\, z_\Delta \chi_\Delta(\xi)$$

where $\chi_\Delta(\xi)$ is the characteristic function of Δ and the numbers $(\gamma_k)_{k=0}^{\infty}$ are given by:

$$\gamma_k = \gamma \qquad k = 0, 1, \ldots \qquad \text{if } d = 1$$

$$g_k = \gamma 2^{k-1} \qquad k = 1, 2, \ldots; \quad \gamma_0 = \gamma \qquad \text{if } d = 2.$$

The covariance of the field $\varphi_\xi^{[\leqslant N]}$ will be denoted:

$$C_{\xi\eta}^{[\leqslant N]} = E(\varphi_\xi^{[\leqslant N]} \varphi_\eta^{[\leqslant N]})$$

To define the « interaction » among the above fields let I be a finite union of tesserae of $\bar{Q}_0$. Denote $(Q_i)_{i=0}^{\infty}$ the pavements of I induced by $(\bar{Q}_i)_{i=0}^{\infty}$ and call P, $P^{(N)}$, $\hat{P}^{(N)}$ the probability distributions of, respectively, the random fields $(z_\Delta)_{\Delta \in \bigcup_{i=0}^{\infty} Q_i}$, $(z_\Delta)_{\Delta \in \bigcup_{i=0}^{N} Q_i}$, $(z_\Delta)_{\Delta \in Q_N}$.

The « interaction » among the fields is described in terms of

$$V_0^{[N]}(\varphi) = -\frac{\lambda}{4!} \int_I :(\varphi_\xi^{[\leqslant N]})^4 : d\xi$$

where, in general, $:x^n:$ is defined for an arbitrary gaussian random variable with dispersion $C = E(x^2)$ as:

$$:x^n: = (\sqrt{2C})^n H_n\left(\frac{x}{\sqrt{2C}}\right) \qquad n = 0, 1, \ldots$$

where H_n is the n-th Hermite polynomial:

$$H_0 = 1\ ; \quad H_1 = x\ ; \quad H_2 = x^2 - (1/2)\ ; \quad H_4 = x^4 - 3x^2 + (3/4)$$

The « ground state energy is defined, if the limits exist, as:

$$E(\lambda) = \lim_{I \to R^{d+1}} \lim_{N \to \infty} |I|^{-1} \log \int (\exp V^{(N)})\, dz$$

where $V^{(N)}$ is just $V_0^{(N)}$ if $d = 1$, while if $d = 2$:

$$V^{(N)} = V_0^{(N)} - \frac{1}{2!} \langle (V_0^{(N)})^2 \rangle_{(2)} - \frac{1}{2!} \langle (V_0^{(N)})^2 \rangle_{(0)} - \frac{1}{3!} \langle (V_0^{(N)})^3 \rangle_{(0)}$$

where:

$$\left\langle \frac{(V_0^{(N)})^2}{2!} \right\rangle_{(0)} = \frac{\lambda^2}{2\cdot 4!} \int\limits_{I\times I} (C_{\xi\eta}^{[\leq N]})^4 \, d\xi \, d\eta$$

$$\left\langle \frac{(V_0^{(N)})^2}{2!} \right\rangle_{(2)} = \frac{\lambda^2}{2\cdot 6} \int\limits_{I\times I} (C_{\xi\eta}^{[\leq N]})^3 \, {:}(\varphi_\xi^{[\leq N]})^2{:} \, d\xi \, d\eta$$

$$\left\langle \frac{(V_0^{(N)})^3}{3!} \right\rangle_{(0)} = -\frac{\lambda^3 3^3}{4^3\cdot 6} \int\limits_{I\times I\times I} (C_{\xi\eta}^{[\leq N]})^2 (C_{\eta\zeta}^{[\leq N]})^2 (C_{\zeta\xi}^{[\leq N]})^2 \, d\xi \, d\eta \, d\zeta \, .$$

The problem of finding upper and lower bounds to $E(\lambda)$ is called in this paper « the ultraviolet stability problem » and the techniques to attack and solve it have been discovered in (2) in the analogous, but slightly more difficult, case of the euclidean field theory, and they have been applied combined with other methods inspired by (2), (3) to the case considered here in (1) under the restriction of large α.

In this work we extend the results of [1], and the related techniques, to the general case (α arbitrary). This is accomplished by essentially reducing the general case to the large α case treated in [1]. The reduction proceeds via some ideas which appear in the work of DINABURG and SINAI on the critical point [4].

The main technical tool will be the theory of the rarefied gases well known in Statistical Mechanics and its relation with the theory of continuous spin systems as discovered by GRUBER and KUNZ, [5, 6], and independently by GLIMM, JAFFE and SPENCER.

2. – Intermezzo on Wick polynomials and gaussian processes.

In this section we recall, for the purpose of later use, some very well known definitions and properties of polynomials related to the theory of gaussian processes [7].

The first well known result that we mention is the « Wick's theorem ».

Let $x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}$ be $p+q$ gaussian random variables, not necessarily independent. Then the integral:

$$E(x_1^{n_1} \ldots x_p^{n} x_{p+1} \ldots x_{p+q})$$

where $n_1, \ldots, n_p$ are integers, can be computed via the following efforts:

1) draw a graph with $p+q$ vertices with open lines emerging from them: from the vertex 1 emerge n_1 lines, from the vertex 2 emerge n_2 lines etc. The lines emerging from a given vertex are regarded as distinct (this can be reminded by attaching a label to each of them).

2) form the set Γ of the graphs obtained from the preceding one by collecting all the lines in pairs and joining the lines of the pairs into single lines (if the total number of lines is odd Γ is empty).

3) if $\gamma \in \Gamma$ and if the lines $\lambda_1, \lambda_2, \ldots, \lambda_k$ of γ connect the pairs of vertices $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$, respectively, assign to γ the « value »:

$$\prod_{l=1}^{k} E(x_{i_l} x_{j_l})$$

4) sum over the possible graphs $\gamma \in \Gamma$ their values: the result is the value of the above expectation.

The formulation of the Wick's theorem can be used to introduce the natural notion of « Wick monomial »:

$$:x_1^{n_1} \ldots x_p^{n_p}:$$

This is a polynomial in $x_1, \ldots, x_p$ whose coefficients are determined by imposing that the rule for computing

$$E(:x_1^{n_1} \ldots x_p^{n_p}: x_{p+1} \ldots x_{p+q})$$

is obtained from the one described before, (for arbitrary q), by modifying the effort 4) into:

4^{W}) sum over all the graphs $\gamma \in \Gamma$ which contain no lines connecting pairs (i, j) of vertices chosen among the first p ones (« graph without self contractions among the vertices $1, 2, \ldots, p$ »).

Such polynomials are uniquely defined once the gaussian integral E is given, and exist. They generalize the definition of Wick power $:x^n:$ and enjoy a remarkable property: if $\sigma_1, \sigma_2, \ldots, \sigma_p$ are real numbers:

$$:\left(\sum_{i=1}^{P} \sigma_i x_i\right)^n: = \sum_{n_1 + \ldots + n_p = n} \frac{n!}{n_1! \ldots n_p!} \sigma_1^{n_1} \sigma_2^{n_2} \ldots \sigma_p^{n_p} :x_1^{n_1} \ldots x_p^{n_p}: .$$

If P is a polynomial:

$$P(x_1, \ldots, x_p) = \sum_{n_1 \ldots n_p} c_{n_1 \ldots n_p} x_1^{n_1} x_2^{n_2} \ldots x_p^{n_p}$$

the symbol $:P(x_1, \ldots, x_p):$ will be used to denote the polynomial:

$$:P(x_1, \ldots, x_p): = \sum_{n_1 \ldots n_p} c_{n_1 \ldots n_p} :x_1^{n_1}, \ldots, x_p^{n_p}:$$

The polynomials of this form are a basis in the linear space of the polynomials in $x_1, \ldots, x_p$ for all choices of the gaussian integral E.

To avoid confusion it would be better to add to the dots a symbol referring to the integral E which is used to construct the Wick polynomials: in the sequel we shall not do this since the Wick polynomials that we shall encounter will be systematically the ones associated with the gaussian measure considered at that moment.

Another interesting property of a gaussian field $(z_\Delta)_{\Delta\in Q_0}$ defined over a finite set Q_0 of indices by a covariance matrix C is that this field can be represented as:

$$z_\Delta = \sum_{\Delta'\in Q_0} (\sqrt{2C})_{\Delta\Delta'}\,\zeta_{\Delta'}$$

where $\sqrt{C}$ is the square root matrix of the matrix C and the $(\zeta_\Delta)_{\Delta\in Q_0}$ are independent gaussian variables with distribution:

$$\prod_{\Delta\in Q_0} (\exp - \zeta_\Delta^2)\frac{d\zeta_\Delta}{\sqrt{\pi}}\,.$$

We shall be interested in using the above representation for the fields $(z_\Delta)_{\Delta\in Q_i}$, $i = 0, 1, 2, \ldots$, introduced in sect. 1.

In that case the matrices $(C^{[N]})_{\Delta\Delta'}$, $\Delta, \Delta'\in Q_0\times Q_0$, have a most remarkable property: given α, β there exist $\theta > 0$ and a function $\gamma(\varrho)$, $\varrho\in(-\infty, \infty)$, such that: $\forall\varrho\in(-\infty, +\infty)$, $\forall\Delta, \Delta'\in Q_0\times Q_0$:

$$\left|(C^{[N]})^\varrho)_{\Delta\Delta'}\right| \leqslant \gamma(\varrho)\exp - 2^N\theta|\xi_\Delta - \xi_{\Delta'}|$$

where $\xi_\Delta =$ centre of Δ.

Again there should be no confusion between the finite matrices $(C^{[N]})$ and the infinite matrix $(\bar{C}^{[N]})$. Their matrix elements with indices $\Delta, \Delta'\in Q_0$ coincide only for $\varrho = 1$.

In the appendix we remind the proof of this property of the second difference operator.

3. – Polymerization of the fields.

In this section we consider a general problem of some intrinsic interest in Statistical Mechanics of continuous spin systems. Its relation to our ultraviolet problem is not immediately manifest to the reader unfamiliar with [1] where it was pointed out: so he may look at the coming setc. 5 to see why it is so interesting for our porposes to attack the problem formulated in the next few lines.

Consider the gaussian random variables $(z_\Delta)_{\Delta\in Q_0}$, previously defined, whose distribution is denoted $\hat{P}^{(0)}$ and whose covariance is $C^{(0)}$.

Let $(g_\Delta)_{\Delta\in Q_0}$, $(g_{\Delta_1\Delta_2})_{\Delta_1,\Delta_2\in Q_0}$, ..., $(g_{\Delta_1,\dots,\Delta_r})_{\Delta_1,\dots,\Delta_r\in Q_0}$ be a family of polynomials in the variables $(z_\Delta)_{\Delta\in Q_0}$, $(z_{\Delta_1}, z_{\Delta_2})_{\Delta_1\Delta_2\in Q_0}$, ..., respectively and consider the function:

$$-H = \sum_{\Delta\in Q_0} :g_\Delta(z_\Delta): + \sum_{\Delta_1,\Delta_2\in Q_0} \delta(\Delta_1, \Delta_2):g_{\Delta_1\Delta_2}(z_{\Delta_1}\, z_{\Delta_2}): + \\ + \dots + \sum_{\Delta_1,\dots,\Delta_r} \delta(\Delta_1, \dots, \Delta_r):g_{\Delta_1\Delta_2,\dots,\Delta_r}(z_{\Delta_1}, \dots, z_{\Delta_r}):$$

where the coefficients δ will be supposed to have the property that there exist A, $\varkappa > 0$ such that:

$$|\delta(\Delta_1, \dots, \Delta_s)| \leqslant A \exp - \varkappa d(\xi_{\Delta_1}, \dots, \xi_{\Delta_s}) \qquad s = 1, 2, \dots, r$$

where ξ_Δ = centre of Δ and, in general, $d(\xi_{\Delta_1}, \dots, \xi_q)$ = length of the smallest connected graph containing $\xi_1, \dots, \xi_q$ = « graph distance between $\xi_1, \dots, \xi_q$ ».

We shall also assume that the maximum degree of the polynomials is D and also that all their coefficients are bounded by A.

Furthermore we shall suppose that the indices of the summations defining H run over sets of mutually distinct tesserae.

The number r will be fixed once and for all togheter with the parameters $\varkappa$ and α and β. In the following, when we shall say that some constant is an « absolute constant » it will mean that it depends only on the values of these fixed parameters and on the dimension d.

We shall consider the following problem: given $B > 0$ consider:

$$\frac{\int (\exp \varepsilon H) \prod_{\Delta\in Q_0} \chi(|\zeta_\Delta| < B)\, \hat{P}^{(0)}(dz)}{\int \prod_{\Delta\in Q_0} \chi(|\zeta_\Delta| < B)\, \hat{P}^{(0)}(dz)}$$

where the variables $(\zeta_\Delta)_{\Delta\in Q_0}$ have been introduced at the end of the preceding sec. 2 and $\chi(|\zeta_\Delta| < B)$ is 1 if $|\zeta_\Delta| < B$ and 0 otherwise.

We want to find sufficient conditions that guarantee that, given an integer k, there is G_k such that the above integral can be written as:

$$\exp\left\{\sum_{i=0}^{k} \frac{\varepsilon^i}{i!} \langle H^i\rangle^T + |I| G_k [(A\, \varepsilon\, D)^{k+1} + 2 \exp - B^2/2]\right\}$$

for some $\tau \in [-1, 1]$, where $\langle H^i\rangle^T$ is defined by:

$$\begin{aligned}
\langle H\rangle^T &= \int H \hat{P}^{(0)}(dz) \\
\langle H^2\rangle^T &= \int H^2 \hat{P}^{(0)}(dz) - \left(\int H \hat{P}^{(0)}(dz)\right)^2 \\
\langle H^3\rangle &= \int H^3 \hat{P}^{(0)}(dz) - 3\left(\int H \hat{P}^{(0)}(dz)\right)^2 \left(\int H \hat{P}^{(0)}(dz)\right) + 2\left(\int H \hat{P}^{(0)}(dz)\right) \\
&\dots\dots\dots\dots
\end{aligned}$$

and ε is small enough.

To find the conditions of validity of the above formula we first eliminate the z variables by expressing them in terms of the ζ variables. The numerator of the expression which we are considering becomes:

$$\int (\exp \varepsilon \tilde{H}) \prod_{\Delta \in Q_0} \chi(|\zeta_\Delta| < B) \exp - \zeta_\Delta^2 \frac{d\zeta_\Delta}{\sqrt{\pi}}$$

where $\tilde{H}$ has the same expression as H with new polynomials $\tilde{g}$ replacing the polynomials g and with new coefficients $\tilde{\delta}$ replacing the coefficients δ.

The linearity of the transformation linking z to ζ implies that the maximum degree of the new polynomials $\tilde{g}$ is still D and, also, the maximum number of tesserae which appear in the indices of the polynomials $\tilde{g}$ is still r.

Furthermore the decay properties of the matrices $(C^{[0]})^{-\frac{1}{2}}_{\Delta\Delta'}$ and $(C^{[0]})^{\frac{1}{2}}_{\Delta\Delta'}$ as $|\xi_\Delta - \xi_{\Delta'}| \to \infty$ imply that we can assume that the coefficients of the polynomials $\tilde{g}$ are all bounded by 1 and, that there exists a constant γ_1 such that:

$$|\tilde{\delta}(\Delta_1, \ldots, \Delta_s)| \leqslant \gamma_1^{-1} A \exp - \gamma_1 \varkappa d(\xi_{\Delta_1}, \ldots, \xi_{\Delta_s})$$

and the constant γ_1 is an absolute constant in the sense specified above.

The above discussion shows that the considered problem is reducible to the case in which the ζ and the z variables coincide.

The above change of variable was inspired by Sinai and reduces the above problem to the one solved in [1] using the Gruber and Kunz polymer theory [5, 6].

In the next section we give a proof of the result that we need for the sake of completeness since we are using here a slightly different notation and, also, because the proof presented here will be somewhat simpler than the one in [1].

Before continuing it is convenient to stress that the above problem could be formulated and solved using ordinary polynomials rather than Wick polynomials and the results would be essentially the same. However, in the application that we have in mind, to the theory of the Markov hierarchical fields, Wick polynomials arise much more naturally and greatly simplify the algebra.

4. – Technical aspects of the polymerization of the fields.

Let $\mathcal{F}$ be the family of the subsets of Q_0 consisting of one, two, ... up to r cubes. Then the function $\tilde{H}$ defined in the previous section can be written as

$$\tilde{H} = \sum_{f \in \mathcal{F}} \tilde{H}_f(\zeta_f)$$

with the self-explaining meaning of the symbols.

Given $B > 0$ we shall use the short-hand notation:

$$\gamma_B(d\zeta) = \chi(|\zeta| < B) \exp - \zeta^2 \frac{d\zeta}{\sqrt{\pi}} \Big/ \int \chi(|s| < B) \exp - s^2 \frac{ds}{\sqrt{\pi}}$$

and consider:

$$Z(\varepsilon) = \int \exp \varepsilon \sum_{f\in\mathcal{F}} \tilde{H}_f(\zeta_f) \prod_{\Delta\in Q_0} \gamma_B(d\zeta_\Delta)$$

which can be transformed as [5, 6]:

$$Z(\varepsilon) = \int \prod_{f\in\mathcal{F}} \exp \varepsilon\tilde{H}_f \prod_{\Delta\in Q_0} \gamma_B(d\zeta_\Delta) \equiv \int \prod_{f\in\mathcal{F}} (\exp \varepsilon\tilde{H}_f - 1 + 1) \prod_{\Delta\in Q_0} \gamma_B(d\zeta_\Delta) \equiv$$
$$\equiv \sum_{f\in\mathcal{F}} \int \prod_{f\in\mathcal{F}} (\exp \varepsilon\tilde{H}_f - 1) \prod_{\Delta\in Q_0} \gamma_B(d\zeta_\Delta) .$$

Examining this expression it appears very natural to define a « polymer » R as a subset of $\mathcal{F}: R = (f_1, f_2, \dots, f_m)$, which is connected in the following sense. Given f_a and f_b in R there is a sequence $i_1 = a, i_2, \dots, i_q = b$ such that f_{i_k} and $f_{i_{k+1}}$ have at least one tessera in common (recall that the f's are sets of up to r tesserae) for $k = 1, 2, \dots, q-1$.

If $R = (f_1, \dots, f_m)$ we define $|R| = m$, and if $\mathfrak{R}$ denotes the family of the polymers we shall say that two polymers $R_1, E_2 \in \mathfrak{R}$ « do not overlap » if:

$$\Big[\bigcup_{f\in R_1} (\bigcup_{\Delta\in f} \Delta)\Big] \cap \Big[\bigcup_{f\in R_2} (\bigcup_{\Delta\in f} \Delta)\Big] = \emptyset$$

and this occurrence will be denoted by $R_1 \cap R_2 = \emptyset$.

With the above notations and definitions we realize that every $F \subset \mathcal{F}$ can be thought as composed of mutually non overlapping polymers and the sum can be written:

$$Z(\varepsilon) = \sum_{\substack{(R_1,\dots,R_p)\in\mathfrak{R} \\ R_i\cap R_j=\emptyset,\, i\neq j}} \prod_{i=1}^{p} \zeta_\varepsilon(R_i)$$

where

$$\zeta_\varepsilon(R) = \int \prod_{f\in R} (\exp \varepsilon\tilde{H}_f - 1) \prod_{\Delta\in R} \gamma_B(d\zeta_\Delta)$$

and $\Delta \in \mathfrak{R}$ means that Δ is one of the cubes consituting one of the f's in R.

Let us denote $\|\tilde{H}_f\| = \max |\tilde{H}_f)$, where the maximum is taken over the ζ's such that $|\zeta_\Delta| \leqslant B$, $\forall\Delta$.

Clearly there exists an absolute constant γ_2 such that:

$$\|\tilde{H}_f\| \leqslant AB^D\gamma_2 \exp - \gamma_1\varkappa d(f)$$

which is implied by the decay properties of the coefficients of the polynomials defining $\tilde{H}$ if $d(f) = d(\xi_{\Delta_1}, \dots, \xi_{\Delta_q})$ when $f = (\Delta_1, \dots, \Delta_q)$.

The Taylor's formula pushed to order k, with remainder of order $k+1$ allows us to deduce that

$$(\exp \varepsilon\tilde{H}_f - 1) = \sum_{n=1}^{k} \frac{\varepsilon^n}{n!}\tilde{H}_f^n + \frac{\varepsilon^{k+1}}{(k+1)!}\tilde{H}_f^{k+1} \exp \tilde{\tau}\varepsilon\tilde{H}_f$$

for some $\tilde{\tau} \in [0, 1]$.

So for $k = 0$ we find:

$$|\zeta_\varepsilon(R)| \leq \prod_{f\in R} (\varepsilon\|\tilde{H}_f\| \exp \varepsilon\|\tilde{H}_f\|)$$

and more generally:

$$\zeta_\varepsilon(R) = \sum_{h=1}^{k} \varepsilon^h \sum_{\substack{\Sigma_f n_f = h \\ n_f \geq 1}} \int \left(\prod_{f\in R} \frac{\tilde{H}_f^{n_f}}{n_f!}\right) \prod_{\Delta\in R} \gamma_B(d\zeta) + \\ + \sum_{h=k+1}^{\infty} \varepsilon^h \sum_{\substack{\Sigma_f n_f = h \\ 1\leq n_f \leq k+1}} \int \prod_{f\in R} \left(\frac{\tilde{H}_f^{n_f}}{n_f!} \exp \tau_f \varepsilon \tilde{H}_f\right) \prod_{\Delta\in R} \gamma_B(d\zeta_\Delta)$$

for some $\tau_f \in [0, 1]$, which implies:

$$\zeta_\varepsilon(R) = \sum_{h=|R|}^{k} \varepsilon^h \zeta^{(h)}(R) + \varepsilon^{k+1} \zeta^{(k+1)}(R;\, \varepsilon)$$

where if E_k is a suitably large absolute constant (e.g. $E_k = ke^k$ would be good enough):

$$|\varepsilon^h \zeta^{(h)}(R)| \leq \prod_{f\in R} (\varepsilon E_k \|\tilde{H}_f\|)$$
$$|\varepsilon^{k+1} \zeta^{(k+1)}(R)| \leq \Big[\prod_{f\in R} (\varepsilon E_k \|\tilde{H}_f\| \exp \varepsilon\|\tilde{H}_f\|)\Big] \cdot (\varepsilon \max_f \|\tilde{H}_f\|)^{\max(0,\, k+1-|R|)}$$

More generally if $\Gamma = (R_1, R_2, \dots, R_p) \in \mathfrak{R}^p$ and if $|\Gamma| = \sum_{i=1}^{p} |R_i|$, and if one puts:

$$\zeta_\varepsilon(\Gamma) \equiv \prod_{i=1}^{p} \zeta_\varepsilon(R_i)$$

it follows that:

$$\zeta_\varepsilon(\Gamma) = \sum_{h=|\Gamma|}^{k} \varepsilon^h \zeta^{(h)}(\Gamma) + \varepsilon^{k+1} \zeta^{(k+1)}(\Gamma,\, \varepsilon)$$

and if $F_k \geq E_k$ is a suitably large absolute constant:

$$|\varepsilon^h \zeta^{(h)}(\Gamma)| \leq \prod_{i=1}^{p} \prod_{f\in R_i} (F_k \varepsilon \|\tilde{H}_f\|)$$
$$|\varepsilon^{k+1} \zeta^{(k+1)}(\Gamma;\, \varepsilon)| \leq \prod_{i=1}^{p} \prod_{f\in R_i} (F_k \varepsilon \|\tilde{H}_f\| \exp \varepsilon\|\tilde{H}_f\|)(\max_f \varepsilon\|\tilde{H}_f\|)^{\max(0,\, k+1-|\Gamma|)}$$

Finally it is very important to remark that the limits:

$$\bar{\zeta}^{(h)}(R) = \lim_{B\to\infty} \zeta^{(h)}(R) \qquad h = 1, \dots, k$$
$$\bar{\zeta}^{(h)}(P) = \lim_{B\to\infty} \zeta^{(h)}(P) \qquad h = 1, \dots, k$$

exist (this is an immediate consequence of the above expressions for the $\zeta^{(h)}(R)$ where, if $h < k + 1$, no exponential appears in the iotegrals); furthermore the decay properties of the $\tilde{\delta}$ coefficients and tbe boundedness of the poefficients of the polynomials in H_f imply the existence of an absolute constants $\bar{F}_h$ such that

$$\varepsilon^h \zeta^{(h)}(\Gamma) = \varepsilon^h \bar{\zeta}^{(h)}(\Gamma) + \exp - B^2/2 \prod_{i=1}^{P} \prod_{f \in R_i} (\bar{F}_h A \varepsilon \exp - \gamma_1 \varkappa d(f))$$

whose strenth is in the absence of a B-dependence in the last product.

We call $\tilde{F}_k = \max_{0 \leqslant h \leqslant k} \max (\bar{F}_h, F_k, \gamma_2)$.

We are now almost ready to formulate and use the main result of the theory of polymers [5, 6, 1].

Let $\mathcal{N}$ be the family of the sets $\Gamma = (R_1, \ldots, R_p) \in \mathfrak{R}^p$, $p = 1, 2, \ldots$ consisting of « completely overlapping polymers »: $\Gamma \in \mathcal{N}$ if, given any two $R_a, R_b \in \Gamma$, there is a chain $i_1 = a, i_2, \ldots, i_s = b$ such that R_{i_l} overlaps with $R_{i_{l+1}}$ in the previously described sense, for $l = 1, 2, \ldots, s - 1$.

Then if $\Delta \in \Gamma$ means that one of the polymers in Γ contains Δ and if diam (Γ) = = [maximal distance among the cubes which appear in the polymers of Γ], the following lemma holds [5, 6, 1]:

LEMMA. – There exists a real function φ^T defined on $\mathcal{N}$ such that:

i) φ^T is translationally invariant (i.e. $\varphi^T(\Gamma) = \varphi^T(\Gamma')$ if Γ' is a translate of Γ).

ii) given $\sigma > 0$ there exists $z_0(\sigma) < 1$ such that: $\forall l \geqslant 0$:

$$\sum_{\substack{\Gamma \ni \Delta \\ \operatorname{diam} \Gamma \geqslant l}} |\varphi^T(\Gamma)| \prod_{R \in \Gamma} \prod_{f \in R} (z_0(\sigma) \exp - \sigma d(f)) < 2^{-l}$$

iii) if $\zeta: \mathfrak{R} \to R$ is such that:

$$|\zeta(R)| \leqslant z_0(\sigma)^{|R|} \prod_{f \in R} \exp - \sigma d(f)$$

then:

$$\sum_{\substack{(R_1, \ldots, R_p) \subset \mathfrak{R} \\ R_i \cap R_j = \emptyset,\, i \neq j}} \prod_{i=1}^{p} \zeta(R_i) = \exp \sum_{p \in \mathcal{N}} \varphi^T(\Gamma) \Big(\prod_{R \in \Gamma} \zeta(R) \Big) .$$

Collecting all the above remarks and using the lemma we can therefore obtain the following theorem (see (1) for some more details):

THEOREM. – Given an integer k and $B > 0$ and assuming:

$$A \tilde{F}_k \varepsilon B^D < \frac{z_0(\gamma_1 \varkappa)}{e}$$

there exists an absolute constant G_k such that:

$$Z(\varepsilon) = \exp\left\{\sum_{h=1}^{k} \frac{\varepsilon^h}{h!} \langle H^h \rangle^T + |I|\bar{\tau} G_k (\exp - B^2/2 + (\varepsilon A^2 B^D)^{k+1})\right\}$$

$$\int \prod_{\Delta \in Q_0} \chi(|\zeta_\Delta| < B) \exp - \zeta_\Delta^2 \frac{d\zeta_\Delta}{\sqrt{\pi}} = \exp |I|\bar{\tau}' G_0 \exp - B^2/2$$

where $\bar{\tau}$, $\bar{\tau}' \in [-1, 1]$.

This is the main technical estimate that we need.

5. – Ground state estimates. Upper bound.

The $d = 1$ case is the simplest and we outline it because it teaches what to do in the $d = 2$ case and, also, opens the way to an estimation of the lower bounds.

With the notations of sect. 2, 1 we can express $V_0^{(N)}$ in terms of the random variables $(z_\Delta)_{\Delta \in Q_N}$ and $X_\Delta = \varphi_\xi^{[\leqslant N]} / \sqrt{2 C_{\xi\xi}^{[\leqslant N]}}$ where $\xi \in \Delta \in Q_N$.

It is useful to first remark a recurrence relation between z_Δ and X_Δ: if $d = 1$:

$$X_\Delta = \frac{z_\Delta + \sqrt{N} X_{\Delta'}}{\sqrt{1 + N}} \qquad \text{if } \Delta \in Q_N, \ \Delta' \in Q_{N-1}, \ \Delta' \supset \Delta$$

while if $d = 2$:

$$X_\Delta = \frac{z_\Delta + X_{\Delta'}}{\sqrt{2}}$$

Then we can write:

$$V_0^{(N)} = -\lambda \sigma_N \sum_{\Delta \in Q_N} (1 + N)^2 H_4 \left(\frac{z_\Delta + \sqrt{N} X_{\Delta'}}{\sqrt{1 + N}} \right)$$

where:

$$\sigma_N = \frac{\gamma 2^{-2N}}{3!}.$$

Then if $B_i = Bi$ for $i \geqslant 1$ and $B_0 = B$ and B is any positive constant (which we take larger than the maximum zero of the fourth Hermite polynomial, say $B > 3$) we see that

$$\int \exp V_0^{(N)} P^{(N)}(dz) \geqslant \int \exp V_0^{(N)} \prod_{i=0}^{N} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i) P^{(N)}(dz).$$

We notice that the decay properties of the matrices $(\sqrt{C^{[N]}})^{\pm 1}$ as functions of the separation between the centres of their indices (which are cubes) discussed at the end of sect. 2 and the relation between the z variables and the ζ variables (cfr. end of sect. 2) imply that when $|\zeta_\Delta| < B_i$, $\forall \Delta \in Q_i$, $\forall i \leqslant N$ the following bound holds:

$$|X_\Delta| \leqslant \gamma_3 \tilde{B}_k \quad \text{if } \Delta \in Q_k\,, \quad k = 0, 1, \ldots, N$$

where $\tilde{B}_k = \sum_{i=0}^{k} B_i$ and γ_3 is an absolute constant.

This remark implies that we can rewrite the function $V_0^{(N)}$ in the form considered in sect. 4 with the following values for the parameters D, r, $\varkappa$, ε, A:

$$D = 4\,; \quad r = 4\,; \quad \varkappa = \frac{\theta}{2}\,; \quad \varepsilon = \sigma_N$$
$$A = 6\lambda(1+N)^2 \tilde{B}_{N-1}^4\,.$$

Hence if $\bar{z}_0 = z_0(\gamma_1\varkappa)$ we see that

$$\sigma_N A \tilde{F}_1 \tilde{B}_{N-1}^4 B_N^4 e < \bar{z}_0$$

for N large enough, say $N \geqslant \bar{N}_0$.

Applying the lemma of the preceding section we see that:

$$\int P^{(N-1)}(dz) \prod_{i=0}^{N-1} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i) \cdot \int \exp V_0^{(N)} \prod_{\Delta \in Q_N} \chi(|\zeta_\Delta| < B_N) \exp - \zeta_\Delta^2 \frac{d\zeta_\Delta}{\sqrt{\pi}} =$$
$$= \int P^{(N-1)}(dz) \prod_{i=0}^{N-1} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i) \left(\exp \int V_0^{(N)} \hat{P}^{(N)}(dz)\right) \cdot$$
$$\cdot \exp |I| 2^{2N} \tau \left(2 \exp - B_N^2/2 + G_1 (6\lambda(1+N)^2 2^{-2N} \tilde{B}_{N-1}^4 B_N^4)^2\right)$$

where $\tau \in [-1, 1]$.

Since:

$$\int V_0^{(N)} P^{(N)}(dz) = V_0^{(N-1)}$$

we realize that the above relation means that:

$$\int P^{(N)}(dz) (\exp V_0^{(N)}) \prod_{i=0}^{N} \prod_{\Delta \in Q_i} \chi(\zeta_\Delta| < B_i) =$$
$$= \exp |I| \sum_{N \geqslant N_0}^{\infty} 2^{2N} \left(2 \exp - B_N^2/2 + G_1 (6\lambda(1+N)^2 2^{-2N} \tilde{B}_{N-1}^4 B_N^4)^2\right) \tau' \cdot$$
$$\cdot \int P^{(N_0)}(dz) (\exp V_0^{(N_0)}) \prod_{i=0}^{N_0} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i) = \exp |I| E_-(\lambda; B) \tilde{\tau}$$

where τ', $\tilde{\tau} \in [-1, 1]$ and:

$$E_- = \sum_{N=0}^{\infty} 2^{2N} \left(2 \exp - B_N^2/2 + (6\lambda(1+N)^2 \tilde{B}_{N-1}^4 B_N^4 2^{-2N})^2\right) + \lambda \sigma_{N_0} (1+N_0)^2 H_0(\tilde{B}_{N_0}) 2^{2N_0}\,.$$

The $d = 2$ case is basically the same as the $d = 1$ case.

We now find that, if $\sigma_N = (2^{-N}\gamma)/3!$, $V^{(N)}$ can be rewritten as:

$$V^{(N)} = -\lambda \sum_{\Delta \in Q_N} [\sigma_N 4H_4(X_\Delta) + \lambda\sigma_N^2 \mu_N(\Delta) 2H_2(X_\Delta) + \lambda\sigma_N^2 \gamma_N(\Delta) - \lambda^2 \sigma_N^3 \gamma'_N(\Delta)]$$

where the coefficients μ_N, ν_N, ν'_N verify the bounds: $\exists \gamma_4$ such that

$$\begin{aligned} &0 \leqslant \nu_N(\Delta) \leqslant \gamma_4 \\ &0 \leqslant \mu_N(\Delta)\,, \quad \nu'_N(\Delta) \leqslant \gamma_4(N+1) \qquad N \geqslant 0 \end{aligned}$$

Proceeding as in the $d = 1$ case we see that:

$$\begin{aligned} \int \exp V^{(N)} P^{(N)}(dz) \geqslant \int \prod_{i=0}^{N-1} \prod_{\Delta \in Q_i} \chi(J_\Delta| < B_i)\, P^{(N-1)}(dz) \\ \int \Bigl(\prod_{\Delta \in Q_N} \chi(|\zeta_\Delta| < B_N) \exp - \zeta_\Delta^2 \frac{d\zeta_\Delta}{\sqrt{\pi}} \Bigr) \exp V^{(N)}\,. \end{aligned}$$

As in the $d = 1$ case it is easy to determine the values of the parameters A, D, r, $\varkappa$, ε:

$$D = 4\,, \qquad r = 4\,, \qquad \varkappa = \frac{\theta}{2}\,, \qquad \varepsilon = \sigma_N\,, \qquad A = 6(\lambda + \lambda^3) N \gamma_4 4 \tilde{B}_{N-1}^4\,.$$

Hence we can apply the theorem of sect. 5, now taking $k = 3$, for N large (say if $N \geqslant \bar{N}_0$). We find:

$$\begin{aligned} \int \Bigl(\prod_\Delta \chi(|\zeta_\Delta| < B_N) \Bigr) \hat{P}^{(N)}(dz)(\exp V^{(N)}) = \exp \Bigl[\langle V^N \rangle^T + \frac{\langle (V^{(N)})^2 \rangle^T}{2!} + \frac{\rangle (V^{(N)})^3 \rangle^T}{3!} \Bigr] \cdot \\ \cdot \exp |I| G_3 \tau \Bigl(2 \exp - B_N^2/2 + (6 \cdot 4 \cdot (\lambda + \lambda^3) N \gamma_4 \tilde{B}_{N-1}^4 \sigma_N)^4 \Bigr) 2^{3N}\,. \end{aligned}$$

The calculation of the gaussian integrals in the $\langle \cdot \rangle^T$ can be explicitely performed and does not offer particular difficulties to the very patient reader [1]: it is in this calculation that the properties of the Wick polynomials discussed in sect. 2 (namely the « Leibnitz formula ») are very useful.

After some labor we obtain that for some constant $\tilde{G}_3$ the r.h.s. of the above expression can be written as:

$$\begin{aligned} \exp |I| \tilde{G}_3 \tau' \Bigl(2 \exp - B_N^2/2 + (6 \cdot 4 \cdot (\lambda + \lambda^3) N \gamma_4 \tilde{B}_{N-1}^4 B_N^4 \sigma_N)^h \Bigr) 2^{3N} \cdot \\ \cdot \exp \Bigl\{ V^{N(-1)} + W^{(N-1)} + E^{(N-1)} - \int V^{(N-1)} W^{(N-1)} P^{(N-1)}(dz) \Bigr\} \end{aligned}$$

where:

$$W^{(N-1)} = 4 \cdot 6^2 \lambda^2 \sigma_N^2 \sum_{(\Delta, \Delta') \in Q_{N-1}} \delta_{\Delta,\Delta'}^{(2)}(N-1) \colon X_\Delta^2 X_{\Delta'}^2 \colon$$

$$E^{(N-1)} = \lambda^2\sigma_N^2 \sum_{\Delta\in Q_{N-1}} \delta_{\Delta_1}^{(2)}(N-1):x_\Delta^6: +$$

$$+ \lambda^2\sigma_N^2 \sum_{\substack{\Delta,\Delta'\in Q_{N-1}\\ \Delta\neq\Delta'}} \sum_{i=1 \text{ or } 3} \bar\delta_{\Delta\Delta'}^{(i)}(N-1)(:X_\Delta^{4-i}X_{\Delta'}^{4-i}: - \delta_{i,3}:X_\Delta^2:) +$$

$$+ \lambda^3\sigma_N^3 \sum_{\Delta\in Q_{N-1}} \sum_{j=1}^{5} \delta_\Delta^{1(j)}(N-1):X_\Delta^{2j}: +$$

$$+ \lambda^3\sigma_N^3 \sum_{\substack{\Delta,\Delta'\in Q_{N-1}\\ \Delta\neq\Delta'}} \sum_{\substack{i+j=2,6\\ i,j>0}} \bar\delta_{\Delta\Delta'}^{1(ij)}(N-1)(:X_\Delta^iX_{\Delta'}^j: - \delta_{i,j}\delta_{i,1}:X_\Delta^2:) +$$

$$+ \lambda^3\sigma_N^2 \sum_{\substack{\Delta,\Delta',\Delta''\in Q_{N-1}\\ \Delta\neq\Delta'\neq\Delta''\neq\Delta}} \sum_{\substack{i+j+k<12\\ i,j,k>0}} \delta_{\Delta\Delta'\Delta''}^{(ijk)}(N-1):X_\Delta^iX_{\Delta'}^jX_{\Delta''}^k:$$

and the properties of the coefficients δ are the following: there exist two absolute constants γ_5, γ_6 such that:

$$|\delta_\Delta^{(j)}(N-1)|\,, \quad |\delta_\Delta^{(j)}(N-1)| \leqslant \gamma_5(1+N)$$

$$|\bar\delta_{\Delta\Delta'}^{(ij)}(N-1)|\,, \quad |\bar\delta_{\Delta\Delta'}^{(ij)}(N-1) \leqslant \gamma_5(1+N)\exp -\varkappa\gamma_6 2^N|\xi_\Delta - \xi_{\Delta'}|$$

$$|\bar\delta_{\Delta\Delta'\Delta''}^{(ijk)}(N-1)| \leqslant \gamma_5(1+N)\exp -\varkappa\gamma_6 2^N d(\xi_\Delta, \xi_{\Delta'}, \xi_{\Delta''})$$

At this point one might become afraid that the expression which we would get out of the integration over the $(N-1)$-order variables is no longer treatable. However it is not so: continuing the integrations does not lead to more complicate expressions.

The simplest way to proceed seems to formally integrate over the variables of order $N-1, \dots, N-k+1$ using the formula given by the theorem of sect. 4 without checking its applicability and, afterwards, proving by induction that the result so obtained is indeed correct.

The inductive hypothesis suggested by this procedure is the following [1]: there exists an absolute constant G such that, supposing $|\zeta_\Delta| < B_l$, $\forall\Delta\in Q_l$, $l = 0, 1, \dots, N-k$:

$$\int (\exp V^{(N)}) \prod_{i=N-k+1}^{N} \Big[\prod_{\Delta\in Q_i} \chi(|\zeta_\Delta| < B_i)\,\hat P^{(i)}(dz)\Big] =$$

$$= \exp \tau''|I|G \sum_{h=N-k+1}^{N} [(\lambda+\lambda^3)^4\sigma_h^4 h^{52}B^{16}2^{3h} + 2\exp -B^2/2\,h^2\,2^{3h}]\cdot$$

$$\cdot\exp\Big\{V^{(N-k)} + \sum_{h=1}^{k}\Big(W^{(N-h,k-h)} - \int V^{(N-k)}W^{(N-h,k-h)}P^{(N-k)}(dz)\Big) + \sum_{Q=1}^{k} E^{(N-h,k-h)}\Big\}$$

where $\tau''\in[-1, 1]$

$$W^{(N-h,k-h)} = \int W^{(N-h)}\hat P^{(N-h)}(dz)\dots\hat P^{(N-k+1)}(dz)$$

and $E^{(N-h,k-h)}$ is given by an expression similar to the one for $E^{(N-h)}$ previously described (in the case $h=1$) with the following substitutions:

i) the coefficients $\bar{\delta}^{\cdots}_{\cdots}(N-h)$ are changed into coefficients $\bar{\delta}^{\cdots}_{\cdots}(N-h, k-h)$ with the same indices. These coefficients verify the relations:

$$|\bar{\delta}^{\cdots}_{\cdots}(N-h, k-h)| \leqslant \gamma_7\big((N-h)+1\big)\left(\exp - \varkappa/2\,\gamma_6 2^{N-h} d(\ldots)\right)\cdot 2^{n/2(k-h)}$$

where n is the degree of the Wick monomial to which $\bar{\delta}^{\cdots}_{\cdots}$ is the coefficient and $d(\ldots)$ is the graph distance among the points which are the centres of the cubes which appear as lower indices. The constant γ_7 can be assumed to be an absolute constant.

ii) the indices appearing in the random variables which are the arguments of the Wick monomial $:x^{n_1}_{\Delta_1}x^{n_2}_{\Delta_2}x^{n_3}_{\Delta_3}:$ appearing in $E^{(N-h)}$ (with Δ_1, Δ_2, $\Delta_3 \in Q_{N-h}$) will be changed into $\bar{\Delta}_1$, $\bar{\Delta}_2$, $\bar{\Delta}_3$ where

$$\bar{\Delta}_i \supset \Delta_i \qquad i=1,2,3 \qquad\qquad \bar{\Delta}_i \in Q_{N-k+1}\,.$$

Actually with some patience one could write down explicit expressions for the above coefficients $\bar{\delta}^{\cdots}_{\cdots}(N-h, k-h)$. This shows that there is no real difference between the $d=1$ and the $d=2$ cases: the $d=1$ case is technically simpler because in that case the calculation can be done to first order while in the $d=2$ case the calculation has to be done to third order.

We shall call $\tilde{V}^{(N-k)}$ the function in curly brackets appearing in the above inductive hypothesis:

$$\tilde{V}^{(N-k)} = V^{(N-k)} + \sum_{h=1}^{k}\left(\left[W^{(N-h,k-h)} - \int W^{(N-h,k-h)}V^{(N-k)}P^{(N-k)}(dz)\right] + E^{(N-h,k-h)}\right).$$

In the next section we shall check the inductive hypothesis.

6. – Checking the inductive hypothesis in the $d=2$ case.

To check the inductive hypothesis we would like to apply the theorem of sect. 4 to perform the $(k+1)$-th step.

Clearly the polynomial definining $\tilde{V}^{(N-k)}$ have the form contemplated in sect. 3, 4. The value of the parameters r, D, $\varkappa$, ε, are now, respectively, 12, 12, $\gamma_6\varkappa$, σ_{N-k}. The only problem is therefore to find a value for A and to show that, in spite of the summation over h appearing in $\tilde{V}^{(N-k)}$, it can be uniformly bounded in N.

To find an estimate for A consider, for instance, the coefficient of $\lambda^2 :X^6_{\Delta}:$ in the sum of the E's ($\bar{\Delta}\in Q_{N-k}$); it is bounded by:

$$\lambda^2\gamma_7\sum_{h=1}^{k}\sum_{\substack{\Delta\subset\bar{\Delta}\\ \Delta\in Q_{N-h}}}\big(1+(N-h)\big)\,\sigma^2_{N-h+1}\frac{1}{2^{3(k-h)}} = \gamma_7\lambda^2\sum_{N-k}^{N} l\sigma_l^2 = O(\sigma^2_{N-k})\,.$$

Similarly the coefficient of $\lambda^2 :X^3_{\bar{\Delta}} X^3_{\bar{\Delta}'}:$ is bounded by:

$$\lambda^2 \gamma_7 \sum_{h=1}^{k} \sum_{\substack{\Delta \subset \bar{\Delta} \\ \Delta' \subset \bar{\Delta}';\Delta,\Delta' \in Q_{N-h}}} (1+(N-h)) \frac{\sigma^2_{N-h+1}}{2^{3(k-h)}} \exp - \varkappa/2\, \gamma_6 d(\xi_\Delta, \xi_{\Delta'}) 2^{(N-h)} = O(\sigma^2_{N-k}(N-k)) .$$

Slightly more complicate is the analysis of the coefficient of $\lambda^2(:X_{\bar{\Delta}} X_{\bar{\Delta}'}: - :X^2_{\bar{\Delta}}:)$ which can be estimated by 0 if $\bar{\Delta} \equiv \bar{\Delta}'$ and, if $\bar{\Delta} \neq \bar{\Delta}'$, by:

$$\text{const}\, \gamma_7 \lambda^2 \sum_{h=1}^{k} (N-h) \sigma^2_{N-h+1} \frac{1}{2^{(k-h)}} 2^{(k-h)} = O(\sigma^2_{N-x}(N-k)) .$$

The remaining coefficients can be estimated in the same way. In this manner one sees that the inductive hypothesis implies that A does not grow beyond some fied constant A_0 times $(\lambda + \lambda^3)\tilde{B}^{12}_{N-k}$. This means that the inductive assumption implies that the induction can be pushed one step forward provided $N-k$ is such that:

$$(\lambda + \lambda^3)\, \bar{F}_3 A_0 \sigma_{N-k} \tilde{B}^{12}_{N-k} B^{12}_{N-k} e < z_0 \left(\gamma_1 \gamma_6 \frac{\varkappa}{2}\right) .$$

Since the inductive hypothesis had been previously formulated by just applying the formula of the theorem of sect. 4 without checking if it was really applicable, it is clear that if we now make one more step the result will be authomatically consistent with the inductive hypothesis with the possible exception of the remainder term; however from the theorem we obtain a rigorous estimate of the remainder at the $(k+1)$-th step and after some long algebra one reaches the conclusion that if G has initially been taken large enough the inductive hypothesis, is valid as long as $N-k$ does not become too small and it is also possible to estimate how small is the N_0 such that when $N-k$ becomes smaller than N_0 the inductive hypothesis is non longer correct. N_0 is not an absolute constant but depends only on λ.

We do not present here the mentioned algebraic calculation.

If we denote:

$$\| \tilde{V}^{(N_0)} \| = \sup_{\zeta, I} \frac{|\tilde{V}^{(N_0)}|}{|I|} \prod_{i=0}^{N_0} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i)$$

the validity of the inductive assumption implies the we have found:

$$\int \exp V^{(N)} \prod_{i=0}^{N} \prod_{\Delta \in Q_i} \chi(|\zeta_\Delta| < B_i)\, P^{(N)}(dz) = \exp \tau E_-(\lambda, B)|I|$$

where $\tau \in [-1, 1]$ and:

$$E_-(\lambda; B) = \| \tilde{V}^{(N_0)} \| + G \sum_{k=0}^{\infty} \left(2 \exp - \tfrac{1}{2} B^2_N + \left(\sigma_k A_0 \tilde{B}^{12}_k B^{12}_k (\lambda + \lambda^3)\right)^4\right) 2^{3k}$$

This concludes the analysis of the upper bound to the ground state energy.

The above result is not immediately applicable to the derivation of a lower bound. We have to make some remarks whose interest will be clear in the next section.

An accurate analisys of the above proof shows that the real property that we have needed of the covariance was that the decay properties mentioned at the end of sect. 2 were uniform in I and N. We could in fact consider a much more general situation in which the box I is not necessarily paved by Q_0 but only by Q_N and, furthermore, the z's of a given order could be allowed to be distributed according to the distribution of a gaussian Ising model with Dirichlet type of boundary conditions on some given tesserae.

The calculation would be essentially the same and, since the decay properties of the covariances would be uniform in the larger class of covariances that we are allowing we would find that the quantity $E(\lambda; B)_-$ could be taken the same for the whole class of models that we are now considering.

We do not discuss the above statements in detail but we shall have to make use of them in the sketch of the proof for the existence of a lower bound to the ground state energy.

7. – Lower bounds. Sketch in the $d = 1,2$ case.

We first remark that the decay properties of the covariances imply that if $\tilde{B}_i = \tilde{B}$ i for $i \geqslant 1$ and $\tilde{B}_0 = \tilde{B}$ then there is a B such that the relations:

$$|X_\Delta| \leqslant \tilde{B} i \quad \text{if } \Delta \in Q_i, \ \forall i = 0, 1, \dots, N$$

imply that:

$$|\zeta_\Delta| \leqslant B_i \quad \text{if } \Delta \in Q_i, \ \forall i = 0, 1, \dots, N$$

This means that the result of sect. 6 implies that:

$$\int \exp V^{(N)} \prod_{i=0}^{N} \prod_{\Delta \in Q_i} \chi(|X_\Delta| < \tilde{B}_i) \leqslant \exp E_-(\lambda; B)|I| .$$

We can now proceed as in [1] and write:

$$\int \exp V^{(N)} P^{(N)}(dz) = \sum_{\substack{D_0, \dots, D_N \\ D_i \subset Q_i, i = 0, \dots, N}} \int \exp V^{(N)} \prod_{i=0}^{N} \Big(\prod_{\Delta \notin D_i} \chi(|X_\Delta| < \tilde{B}_i) \prod_{\Delta \notin D_i} \chi(|X_\Delta| > \tilde{B}_i) \Big) P^{(N)}(dz) .$$

Let us examine a single term of the above sum and call:

$$\bar{\chi}^{(N)} = \prod_{i=0}^{N} \Big(\prod_{\Delta \notin D_i} \chi(|X_\Delta| < \tilde{B}_i) \prod_{\Delta \in D_i} \chi(|X_\Delta| > \tilde{B}_i) \Big) .$$

It is natural to consider the regions:

$$\mathfrak{D}_N = \bigcup_{\Delta \in D_N} \Delta\,, \qquad \mathfrak{D}_{N-1} = \bigcup_{\Delta \in D_{N-1}} \Delta\,, \qquad \mathfrak{D} = \mathfrak{D}_N \cup \mathfrak{D}_{N-1}\,.$$

We now start integrating with respect to the N-th order variable associated with the tesserae $\Delta \not\subset \mathfrak{D}$ after fixing the values of the z-variables withi ndices contained in $\mathfrak{D}$ as well as those which are not in $\mathfrak{D}$ but have a nearest neighbour which is in $\mathfrak{D}$. The union of the tesserae of this last type will be abridged in ∂M and the meaning of the symbol ∂M will make sense in the natural way also for more general sets.

The integration can be performed basically in the same way we did in the case treated in the last section. There are of course some obvious modifications: in fact the conditional measure with respect to which we are integrating is a non centred gaussian measure with the covariance associated to the gaussian Ising model with zero boundary conditions on ∂M (cfr. appendix). The already mentioned properties od decay of the covariances imply that the gaussian variables z_Δ are centred at a point which tends to zero with exponential speed $(\exp - \theta 2^N d(\xi_\Delta, \partial M))$ when ξ_Δ goes far from the boundary ∂M; also the new covariance differs from the old one only near the ∂M and the difference between two matrix elements with the same indices tends exponentially to zero on the scale 2^{-N} as the minimum distance from the indices to the boundary ∂M tend to grow. Another minor difficulty is due to the fact that now our functions are no longer naturally written as Wick polynomials, the covariance having changed, again they are «almost» Wick polynomials except when they contain random variables with indices close to the boundary.

These facts imply that the result of the application of the theorem of sect. 4 will be, cfr. also the remaks at the end of sect. 5:

$$(*) \quad \bar{\chi}^{(N)}\Big(\exp \sum_{\substack{\Delta \in Q_N \\ \Delta \subset \mathfrak{D}}} V_\Delta^{(N)}\Big)(\exp \tilde{V}_{\mathfrak{D}}^{(N-1)})\big(\exp 2^{3N}|\mathfrak{D}|O(\sigma_N \lambda B_N^4)\big)$$
$$\exp G 2^{3N}|I|\big(2 \exp - B/2_N^2 + (\sigma_N A_0 \tilde{B}_N^{12} B_N^{12}(\lambda + \lambda^3))^4\big)$$

where p is some high power, $\tilde{V}_{\mathfrak{D}}^{(N-1)}$ is the expression which would have been obtained starting from a I of the shape $I \setminus \mathfrak{D}$ and with a free field in which the variables with index N were distributed with a zero boundary condition on ∂M; the $V_\Delta^{(N)}$ is:

$$-\lambda\sigma_N 4 H_4(X_\Delta) - \lambda^2 \sigma_N^2 \mu_N(\Delta) 2 H_2(X_\Delta) - \lambda^2 \sigma_N^2 \nu_N(\Delta) + \lambda^3 \nu_N'(\Delta)\,.$$

The next step is to say that $(*)$ can be majorized by:

$$\bar{\chi}^{(N)}(\exp \tilde{V}_{\mathfrak{D}_{N-1}}^{(N-1)})\Big(\exp - \lambda\sigma_N \sum_{\substack{\Delta \in \mathfrak{D}_{N-1} \setminus \mathfrak{D}_N \\ \Delta \in Q_N}} V_\Delta^{(N)}\Big)\big(\exp - |\mathfrak{D}_N| 2^{3N} \lambda \sigma_N (B_N^4/2)\big) \cdot$$
$$\cdot \big(\exp 2^{3N} |\mathfrak{D}_{N-1} \setminus \mathfrak{D}_N| O(\lambda\sigma_N)\big)\Big(\exp G 2^{3N}|I|\big(2 \exp - B_N^2/2 + (\lambda + \lambda^3)^4 (\sigma_N B_N^{12} \tilde{B}_N^{12})^4\big)\Big)\,.$$

This bold step is made possible using the positivity properties of the Hermite polynomials which allow to make a very strong majorization in the regions where the fields are high.

Since we shall majorize the remainder terms by their maxima we see that the integration over the conditions involves only the characteristic functions relative to the tesserae inside $\mathfrak{D}$. The integral:

$$\int \exp - \lambda \sum_{\Delta \notin \mathfrak{D}_{N-1} \setminus \mathfrak{D}_N} V_\Delta^{(N)} \bar{\chi}^{(N)} P^{(N)}(dz)$$

can be estimated by:

$$(1 + \exp - B_N^2/4)^{2^{3N}|I|} \int \bar{\chi}^{(N)} P^{(N)}(dz)$$

and it is now possible to perform the sum over the choices of D_N: the result is:

$$\int \exp V^{(N)} P^{(N)}(dz) \leqslant \sum_{D_N \dots D_{N-1}} \int \bar{\chi}^{(N-1)} \exp \tilde{V}^{(N-1)} \cdot \mathfrak{D}_{N-1} \cdot$$
$$\cdot \exp G 2^{3N} |I| \big(2 \exp - B_N^2 + (\sigma_N A_0(\lambda + \lambda^3) B_N^{12} \tilde{B}_N^{12})^4\big) \,.$$

The procedure can be easily iterated and therefore one arrives at the lower bound:

$$E_+(\lambda; B) = E_-(\lambda, B) + \sum_{N=0}^{\infty} 2^{3N} \exp - B_N^2/4 + O(\sigma_N N^p) \,.$$

Appendix: Covariances and their properties.

Let M be a union of Δ's in $\bar{Q}_0$ and call $\bar{\partial} M$ the set: $\{\xi | \exists \Delta \ni \xi, \ \Delta \in \bar{Q}_0, \ \Delta \notin M\}$.

Consider a gaussian field of Ising type with zero boundary conditions on $\bar{\partial} M$, with formal density

$$\text{const} \exp - \frac{\beta}{2} \Big(\sum_{(\Delta, \Delta') \in M \cup M} (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_{\Delta \in M} z_\Delta^2 \Big)$$

where the first sum runs over the nearest neighbour pairs in $M \cup \partial M$ and $z_\Delta \equiv 0$ if $\Delta \subset \partial M$.

This field is the gaussian field with covariance:

$$\bar{C} = \beta^{-1} (\alpha^2 - D_{(\partial M)})^{-1}$$

where $D_{(\partial M)}$ is the second difference operator with Dirichlet boundary conditions on ∂M; $\bar{C}$ is a « $M \times M$ » matrix.

Let $I \subset M$ be a set which is exactly paved by Q_0, then we call C the « $I \times I$ » matrix:

$$C_{\Delta\Delta'} \equiv \bar{C}_{\Delta\Delta'} \quad \text{if } \Delta, \Delta' \in I$$

We are interested in studying properties of the matrices $\bar{C}^{\varrho}$, C^{ϱ} when $\varrho \in (-\infty, \infty)$.

Let us introduce the following auxiliary matrices A^i, A^e with indices in $I \times I$ and $(M/I) \times (M/I)$ respectively: they are the matrices generated by the quadratic forms:

$$\beta\Big(\sum_{(\Delta,\Delta')\in I\times I} (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_\Delta z_\Delta^2\Big)$$

and respectively:

$$\beta\Big(\sum_{\Delta,\Delta')\in(M\setminus I)\times(M\setminus I)} (z_\Delta - z_{\Delta'})^2 + \alpha^2 \sum_\Delta z_\Delta^2\Big)\,.$$

Introduce also the maps between R^I and $R^{(M/I)}$:

$$(Ez)_{\Delta'} = 2 \sum_{\substack{\Delta\in I \\ \Delta \text{ nearest neigh. } \Delta'}} z_\Delta \qquad \Delta' \in M/I$$

and its adjoint:

$$(E^* z)_\Delta = 2 \sum_{\substack{\Delta'\in I \\ \Delta \text{ nearest neigh. } \Delta'}} z'_{\Delta'} \qquad \Delta \in I$$

Then it is easy to see that:

$$C^{-1} = A^i - \tfrac{1}{4} E^* (A^e)^{-1} E$$

Finally let (Δ_0, t_0), (Δ_1, t_1), ..., (Δ_k, t_k) be « a Brownian path on Q_0 » which starts at Δ_0 and stays there a time t_0, then jumps to Δ_1 and stays there a time t_1 etc. If ω denotes the above path define the time spent on the boundary by ω as:

$$f(\omega) = \sum_{j=0}^{k} n_I(\Delta_j)\, t_i$$

where $n_I(\Delta)$ is the number of neighbours of Δ which are not in I.

If $W_{\Delta\Delta'}$ is defined as

$$W_{\Delta\Delta'} = \beta\delta_{|\xi_\Delta - \xi^*,1} + \beta\tfrac{1}{4}(E^* A^e E)_{\Delta\Delta'}$$

where $\delta_{x,1}$ is zero if x and 1 are not nearest neighbours.

Then our starting point will be the representation:

$$(\exp - (C^{-1} - \zeta)\, t)_{\Delta\Delta'} = \sum_{k=0}^{\infty} \sum_{\Delta_0=\Delta,\dots,\Delta_k=\Delta'} \int \prod_{i=0}^{k} dt_i\, \delta(\Sigma_i t_i - t) \prod_{i=0}^{k-1} W_{\Delta_i,\Delta_{i+1}} \exp - (2(d+1)\beta + \alpha^2\beta + \zeta t)$$

valid for $\operatorname{Re}\zeta < 0$.

The $\bar{C}$ has the simpler representation:

$$(\exp - \bar{C}t)_{\Delta\Delta'} = \sum_{k=0}^{\infty} \frac{N_k^{(\partial M)}(\Delta, \Delta')}{k!} t^k \exp - (2(d+1+\alpha^2)\, t\,.$$

Where $N_k^{(\partial M)}(\Delta, \Delta')$ = number of paths with nearest neighbour jumps joining Δ to Δ' and not entering (∂M).

These formulae imply that:

$$(*) \qquad \left|(\exp - (C^{-1} - \zeta)\, t)_{\Delta\Delta'}\right| \leqslant (\exp - \bar{C}^{-1} t)_{\Delta\Delta'} \qquad \operatorname{Re} \zeta \leqslant 0$$

$$(**) \qquad \left|(\exp - (\bar{C}^{-1} - \zeta)\, t)_{\Delta\Delta'}\right| \leqslant (\exp - \bar{C}^{-1} t)_{\Delta\Delta'} \qquad \operatorname{Re} \zeta < 0$$

which imply integrating over t:

$$\left|\left(\frac{1}{C^{-1} - \zeta}\right)_{\Delta\Delta'}\right| \leqslant \bar{C}_{\Delta\Delta'} \qquad \operatorname{Re} \zeta < 0$$

$$\left|\left(\frac{1}{\bar{C}^{-1} - \zeta}\right)_{\Delta\Delta'}\right| \leqslant \bar{C}_{\Delta\Delta'} \qquad \operatorname{Re} \zeta < 0\,.$$

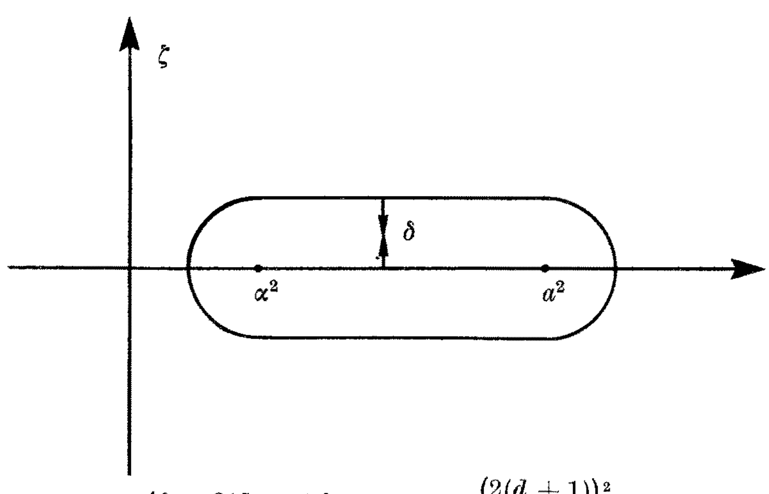
Next observe that the brownian motion representation for $\bar{C}$ implies that the covariance with Dirichlet Boundary conditions on some set is such that every matrix elements of it is bounded by the matrix element with equal indices of the covariance without boundary conditions which is well known to decay as the distance among the indices grows, exponentially fast.

Hence we deduce from the above that there exist constants θ', θ such that for all M, I:

$$|\bar{C}^{-1} - \zeta|_{\Delta\Delta'} \leqslant \theta' \exp - \theta|\xi_\Delta - \xi_{\Delta'}|$$

$$|C^{-1} - \zeta)_{\Delta\Delta'} \leqslant \theta' \exp - \theta|\xi_\Delta - \xi_{\Delta'}|$$

Let now Γ be a contour as drawn in the picture:



$$A^2 = 3(d+1)\beta\alpha^2 + \alpha^2 + 4\,\frac{(2(d+1))^2}{\alpha^2}\,.$$

Then:

$$(C^{-\varrho})_{\Delta\Delta'} = \frac{1}{2\pi i}\int\left(\frac{z^{\varrho}\,dz}{C^{-1}-z}\right)_{\Delta\Delta'} \qquad \varrho \in (-\infty, +\infty)$$

and a similar formula holds for $\bar{C}^{\varrho}$. This is due to the choice of A^2 which is large enough so that the spectrum of every matrix C and $\bar{C}$ for all I and M is in the interval $[\alpha^2, A^2]$.

By the spectral theorem on selfadjoint operators the functions

$$f_{\Delta\Delta'}(\zeta) = \left(\frac{1}{C^{-1}-\zeta}\right)_{\Delta\Delta'}, \qquad \bar{f}_{\Delta\Delta'}(\zeta) = \left(\frac{1}{\bar{C}-\zeta}\right)_{\Delta\Delta'}$$

are uniformly bounded is a open region containing Γ and connected to the left half plane.

By the Riemann mapping theorem there is a holomorfic function which maps this open region into the unit circle and sends to the origin a chosen point $\bar{z}$ in the region. Consider such a map $\xi = \xi(z)$ and then write the functions $f_{\Delta\Delta'}(\xi)$ and $\bar{f}_{\Delta\Delta'}(\xi)$ in terms of their Taylor series around the origin in ξ plane.

We can estimate the coefficients using the Cauchy formula taking as integration contour either a small circle around the origin which we can assume so small that on it:

$$|f_{\Delta,\Delta'}(\xi)|, \qquad |f_{\Delta\Delta'}(\xi)| \leqslant \theta' \exp - \theta'' d(\Delta, \Delta')$$

or we can take a large circle, so large as to contain in its interior the image of the contour Γ. Using the equiboundedness it follows that we can choose between two bounds for the Taylor series coefficients of the functions that we are studying. If me choose one or the other possibility in order to minimize the sume of the absolute values of the terms of the two power series we find that there are on $\hat{\theta}$, $\hat{\theta}'$ such that for all I, M:

$$|(C^{\varrho})_{\Delta\Delta'}|, \qquad |\bar{C}^{\varrho}_{\Delta\Delta'}| \leqslant \hat{\theta}' \exp - \hat{\theta} d(\Delta, \Delta')$$

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