Physica **70** (1973) 219–221 © *North-Holland Publishing Co.*

SOME REMARKS ON ISING-SPIN SYSTEMS⁺

G. GALLAVOTTI and J. L. LEBOWITZ‡

Belfer Graduate School of Science, Yeshiva University, New York, N.Y. 10033, USA

Received 18 Juni 1973

Synopsis

We show that: (a) the free energy and correlation functions of the two-dimensional Ising-spin system with nearest-neighbour ferromagnetic interactions, remain infinitely differentiable with respect to β and h as $h \to 0^{\pm}$ for $\beta > \beta_c$ (where β_c is the reciprocal of the critical temperature) and, (b) the equilibrium equations for the correlation functions of Ising-spin systems may admit a non-physical solution even in the region, $\beta < \beta_c$, where they are known to have a unique physical solution.

1. *Proof of (a).* Consider an Ising-spin system with ferromagnetic pair interactions in a domain $\Lambda \subset Z^{\nu}$. We shall denote by '+' the boundary condition in which all spins in $Z^{\nu}\backslash\Lambda$ are +1. Let $u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h, \Lambda, +)$ be the pair correlation: $\langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle\langle\sigma_y\rangle$ for this system, $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$. The argument used in ref. 1 (employing the Griffiths, Hurst and Sherman inequality[2]), then shows that when the magnetic field h is in the up direction then

$$u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h, \Lambda, +) \le u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h, +) \le u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h = 0, +), \tag{1}$$

where $u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h, +) = \lim_{\Lambda\to\infty} u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h, \Lambda, +)$, the limit being approached monotonically.

We now observe that for the two-dimensional system, $\nu = 2$, with nearest-neighbour attractive interactions, it was shown in ref. 4 that in the infinite-volume limit $\langle\sigma_x\sigma_y\rangle_+(\beta) = \langle\sigma_x\sigma_y\rangle_p(\beta)$; here p indicates periodic (or cylindrical) boundary conditions, and the equality holds for all β even when $h = 0$. (For $h \ne 0$ or $\beta < \beta_c$, the result was already known before[3].) Furthermore in ref. 4 it is also shown that $\lim_{|x-y|\to\infty}\langle\sigma_x\sigma_y\rangle_p = \langle\sigma_x\rangle_+^2$. It follows then from the explicit compu-

⁺ Supported in part by the A.F.O.S.R. Grant #73-2420.

‡ Permanent address: Istituto di Fisica Teorica dell'Università Mostra d'Oltremare pad. 19 – 80125 Napoli. Lavoro eseguito nell'ambito di un programma presentato al C.N.R. (G.N.F.M.).

tation of Wu[5]) that the right side of (1) has an exponential decay[1]):

$$u_2(\boldsymbol{x}, \boldsymbol{y}; \beta, h = 0, +) \leq \text{const.} \exp -K|\boldsymbol{x} - \boldsymbol{y}| \tag{2}$$

with $K > 0$ for $\beta > \beta_c$. This in turn implies infinite differentiability by the arguments given in ref. 1. (We note here that Martin-Löf obtained the bound (2) by direct computation and communicated it to us prior to our result.)

Actually in ref. 5 the author deals with the case when $\boldsymbol{x}$ and $\boldsymbol{y}$ are on the same horizontal or vertical line; the general case follows from a careful examination of the spectrum of the transfer matrix and it is particularly easy to obtain if one is content with a weak estimation of the form

$$|\langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle_+^2| \leq \text{const.} \exp -\tfrac{1}{2}K|\boldsymbol{x} - \boldsymbol{y}|,$$

where K is the horizontal or vertical correlation length.

2. *Proof of (b).* To prove (b) we consider a one-dimensional system with nearest-neighbour interaction, $\Lambda = [-L, L]$, and 'open' boundary conditions corresponding to no interactions with spins outside Λ. The hamiltonian of this system, for $h = 0$, then is $H_0(\sigma) = -\sum_{L \equiv -1}^{L-1} \sigma_i\sigma_{i+1}$. Let $H_i(\sigma) = H_0(\sigma) - (\mathrm{i}\pi/2\beta) \times (\sigma_{-L} + \sigma_L)$. We shall denote with a subscript $L, 0$ or L, i the average obtained by using $\mathrm{e}^{-\beta H_0}$ or $\mathrm{e}^{-\beta H_i}$ as weights and by a subscript 0 or i we shall mean the limit as $L \to \infty$, of the corresponding quantities with subscript $L, 0$ or L, i.

If $x_1 < x_2 < \cdots < x_m$ a simple computation leads to the following result

$$\langle\sigma_{x_1} \cdots \sigma_{x_{2n+1}}\rangle_i = 0 = \langle\sigma_{x_1} \cdots \sigma_{x_{2n+1}}\rangle_0, \tag{3}$$

$$\langle\sigma_{x_1} \cdots \sigma_{x_{2n}}\rangle_i = \prod_{j=1}^{2n-1} \langle\sigma_{x_j}\sigma_{x_{j+1}}\rangle_i,$$

$$\langle\sigma_{x_1} \cdots \sigma_{x_{2n}}\rangle_0 = \prod_{j=i}^{2n-1} \langle\sigma_{x_j}\sigma_{x_{j+1}}\rangle_0. \tag{4}$$

Furthermore it is easy to check that:

$$\langle\sigma_x\sigma_y\rangle_i \equiv \lim_{L\to\infty} \frac{\langle\sigma_x\sigma_y\sigma_{-L}\sigma_L\rangle_{0,L}}{\langle\sigma_{-L}\sigma_L\rangle_{0,L}} = \frac{1}{\langle\sigma_x\sigma_y\rangle_0}. \tag{5}$$

hence $\langle\sigma_x\sigma_y\rangle_i > 1$ and therefore $\langle\sigma_x\sigma_y\rangle_i$ cannot correspond to a physically acceptable state. It is, however, easy to see from the definition

$$\langle\sigma_x\sigma_y \cdots\rangle_i = \lim_{L\to\infty} \frac{\sum_\sigma (\sigma_x\sigma_y \cdots) \exp -\beta H_i(\boldsymbol{\sigma})}{\sum_\sigma \exp -\beta H_i(\boldsymbol{\sigma})} \tag{6}$$

that the $\langle\sigma_X\rangle_i$ define a family of local distributions $f_\Lambda(X)^{+}$ which verify the equilibrium equations (6) as well as the compatibility and normalization conditions (7) that they would have to satisfy if they came from a probability measure on the space of the spin configurations.

Notice that, since $\langle\sigma_x\sigma_y\rangle_0\,(\beta) = (\text{th}\,\beta)^{|x-y|}$ the functions $\langle\sigma_x\sigma_y\rangle_i\,(\beta)$ are singular around $\beta = 0$ which explains why they cannot be obtained by the usual perturbative expansions around $\beta = 0$.

It could be directly checked that the Kirkwood–Salsburg equations in zero field are, in general, invariant under the transformation $J_{ij} \to J_{ij} + \mathrm{i}\pi/2\beta$ (this is because only $\exp(-4\beta J_{ij})$ enters into the KS equations) and this remark, applied to our case, could be used to provide a simple direct proof that the correlation functions $\langle\sigma_X\rangle_i$ are a solution to the KS equations [one merely notices that $\text{th}\,(\beta + \frac{1}{2}\mathrm{i}\pi) = 1/\,\text{th}\,\beta$].

REFERENCES

1) Lebowitz, J.L., Commun. math. Phys. **28** (1972) 313.
2) Griffiths, R.B., Hurst, C.A. and Sherman, S., J. math. Phys. **11** (1970) 790.
3) Lebowitz, J.L. and Martin-Löf, A., Commun. math. Phys. **25** (1973) 272
4) Benethin, P., Gallavotti, G., Jona, G. and Stella, A., Commun. math. Phys. **30** (1973) 45.
5) Wu, T.T., Phys. Rev. **149** (1965) 380.
6) Lanford, O. and Ruelle, D., Commun. math. Phys. **13** (1969) 194.
7) Ruelle, D., Statistical Mechanics, Benjamin (New York, 1969) p. 169.

+ If $X = (x_1, x_2, \ldots, x_p)$ the functions $f_\Lambda(X)$ are the „probabilities" for finding, inside Λ, spins up in the points $x_1 \ldots x_p$ and spins down in the remaning points; *i.e.*,

$$f_\Lambda(X) = \left\langle \prod_{\xi\in X} \left(\frac{\sigma + 1}{2}\right) \prod_{\xi\in\Lambda/x} \left(\frac{1 - \sigma_\xi}{2}\right)\right\rangle_i. \quad \text{Also } \langle\sigma_X\rangle = \left\langle \prod_{i=1}^{p} \sigma_{x_i}\right\rangle.$$