Some problems connected with the phase separation in the Ising model at low temperature

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Abstract: *In the following we are going to give an account of some recent results describing a system in which two phases can coexist, starting from the basic assumptions of Statistical Mechanics.*

Introduction

We will consider the 2-dimensional Ising model with nearest neighbour interaction and without external magnetic field , this being the "simplest" model which can be shown to undergo a phase transition at low temperature. This transition is manifested *e.g.* by the instability of the average magnetization with respect to perturbations in the boundary conditions or in the external field.

For example, if the boundary spins are all $+1$ (-1) then the average magnetization of an infinite system will be $+m^*$ $(-m^*)$, $m^* > 0$ being the spontaneous magnetization. We are going to use the terminology of Dobrushin, [1], and Lanford-Ruelle, [2], (applicable to much more general spin systems) and call Gibbs-state or equilibrium state of a finite system a probability distribution for its configuraticns defined by the Boltzmann factor and by fixing some boundary condition. Such a probability distribution is characterized by specifying the correlation functions $\langle\sigma_{x_1}\ldots\sigma_{x_n}\rangle$ for all finite families $\{\sigma_{x_1},\ldots,\sigma_{x_n}\}$ of spins on the lattice.

A Gibbs state of an infinite system is then defined as a probability distribution, or a family of correlation functions, which are limits of correlation functions of an increasing in family of systems with some boundary conditions.

The occurrence of a phase transition in the above sense then reveals itself by the existence of more than one Gibbs-state for the infinite system; the correlation functions can have different limits depending on the boundary conditions. The set of possible Gibbs-states for the infinite system can be seen to form a convex compact set of probability distributions in a suitable topology, and an arbitrary Gibbs-state can be represented as a convex linear combination ("mixture") of extremal (in the sense of

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convexity theory) Gibbs-states. Such extremal states are often identified with "pure phases". They have correlations which decay over large distances: $\langle\sigma_{x_1}\dots\sigma_{x_n}\sigma_{y_1}\dots\sigma_{y_n}\rangle \xrightarrow[d\to\infty]{} \langle\sigma_{x_1}\dots\sigma_{x_n}\rangle\langle\sigma_{y_1}\dots\sigma_{y_n}\rangle$ as the distance $d = d(x_1,\dots,x_n;y_1,\dots,y_n) \to \infty$ at least in the Cesaro sense.

This somewhat abstract definition of "pure phase" is elaborated in Section 2 for the model we consider. It is shown that at low temperature there are only two extremal translationally invariant Gibbs-states for the infinite system namely those obtained by taking as boundary conditions: all spins at the boundary equal to $+1$ or -1. These states have average magnetization $\pm m^*$ respectively and have decaying correlations. An arbitrary translationally invariant Gibbs-state is then a "mixture" of these two states in the sense that its correlation functions are given by $\langle\sigma_{x_1}\dots\sigma_{x_n}\rangle = \alpha\langle\sigma_{x_1}\dots\sigma_{x_n}\rangle_+ + (1-\alpha)\langle\sigma_{x_1}\dots\sigma_{x_n}\rangle$ for some α with $0 \le \alpha \le 1$.

From the proof of the above relation it is furthermore seen that it is a consequence of the fact that in a large finite system with some given boundary condition a typical configuration can be described as a mixture (in the ordinary sense) of large regions, in which the state is described by either of the two extremal probability distributions; the average proportions of the $+$ regions and the $-$ regions being approximately α and $(1-\alpha)$ respectively. "Large" in the above picture means that the parts of the region near the boundaries separating $+$ and $-$ regions, where boundary effects are noticeable, is negligible.

This means that the correlation functions $\langle\sigma_{x_1}\dots\sigma_{x_n}\rangle$ above can be interpreted as describing the state of a family $\{\sigma_{x_1+a},\dots,\sigma_{x_n+a}\}$ with a chosen "at random" in the box containing the system, because with probability α the family will fall well inside a $+$ region and with probability $1-\alpha$ well inside a $-$ region.

In Section 3,5, and 6 we study a situation in which the separation of the two phases can be investigated and the surface tension coming from their "surface" of separation can be exhibited. A simple boundary condition producing a separation of the two phases is the one we consider: the lattice is periodic in the horizontal direction (i.e. it is a vertical cylinder) and it has the boundary condition $+$ on the top edge and $-$ on the bottom edge.

We show in Section 3 and 5 that if one considers the canonical ensemble with a fixed magnetization $m = \alpha m^* + (1-\alpha)(-m^*)$ and these boundary conditions, then a typical configuration will consist of one $+$ region on top of one $-$ region. In these regions the average magnetizations are very nearly $+m^*$ and $-m^*$ respectively, and the proportions of the areas are very nearly α and $(1-\alpha)$. Furthermore the border going around the cylinder, which separates the two regions, has a length which does not exceed very much the circumference of the cylinder.

In Section 3 and 6 we show that there is a surface tension associated with this border, which does not depend on α, and which therefore can be called the surface tension between two coexisting phases (to be distinguished from the surface tension between the fixed spins at the top, *e.g.*, and the + phase below).

In Section 4 we develop a "cluster theory" of the two pure phases, which is a basic tool in the proofs of Section 5 and 6. In Section 7 we discuss some problems related to those treated in the preceding sections. Some technical proofs are in the appendices: Appendix 1 is for Section 2 and Appendix 2 is for Section 5.

The results of Section 2 are due to Gallavotti, Miracle-Sole [3]. The results and proofs concerning the phase separation in Sections 3 and 5 are basically those of Minlos and Sinai, [4, 5]. However, their treatment has been simplified at several points where we make use of the cylindrical boundary conditions, which they do not consider, and of the cluster theory of Section 3, the use of which in many respects shortens their proofs considerably. The results concerning the surface tension are due to Gallavotti and Martin-Löf, [6], and the proofs given here are small modifications of the same in [6].

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1 Notations and definitions

Let θ be a finite subset of the infinite square lattice $\mathbb{Z}^Z2$. We recall that the Ising model in the "box" θ is defined by associating to each point $x \in \theta$ a spin variable σ_x taking values ± 1. We shall always suppose that the spins on the boundary of 0 are fixed, and we denote by $\boldsymbol{\tau}$ the array specifying their values. To each configuration of spins in θ, having the specified boundary values $\boldsymbol{\tau}$, we associate the weight (Boltzmann-factor):

$$\widetilde{w}_{\boldsymbol{\tau}}(\boldsymbol{\sigma}) = e^{\frac{1}{2}\sum_{x,y}\sigma_x\sigma_y}, \tag{1.1}$$

where the sum runs over all pairs of nearest neighbors in θ (including the boundary spins), and β is proportional to the inverse temperature. The probability of an allowed spin configuration is then determined by the normalized weight.

For our purpose of investigating the system at low temperature, (i.e. for β large) it will be very convenient to represent a configuration not as an array $\boldsymbol{\sigma}$ but by specifying the boundaries separating $+$ and $-$ spins. This can be done as follows. Given a configuration $\boldsymbol{\sigma}$, draw for each bond on the lattice having opposite spins at its endpoints a segment of length one perpendicular to the bond and centered at its midpoint. In this way we obtain a family of lines on the lattice shifted from the original one. The segments separating boundary spins are fixed by $\boldsymbol{\tau}$; they and the others form a graph such that each point on the shifted lattice "inside" θ belongs to $0, 2$, or 4 segments. The last case occurs if the spins around the point are arranged as

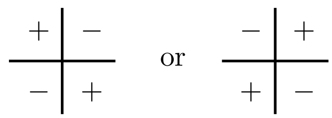


Figure 1

If in this case we modify the lines as follows

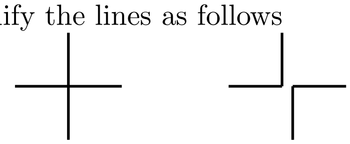


Figure 2

we realize that the set of lines splits into a family of edge-selfavoiding contours, some of which start and end at the fixed segments of the boundary, the others being closed and lying "inside" θ, (the contours are allowed to touch as in Fig. 2 but not in any other way). We denote contours by the letter γ and their lengths by $|\gamma|$ and also the number of points in θ by $|\theta|$.

Often we will denote open contours that start that start and end at the boundary by the letter η. Any closed contour has at least length 4 and any open one at east the length 2. We will often denote an equivalence class of congruent contours by (γ). Given the boundary condition $\boldsymbol{\tau}$ a spin configuration is uniquely characterized by its associated family of contours, so we can talk about contour configurations instead of spin configurations without ambiguity. The weight of a configuration $(\gamma_1, \ldots, \gamma_n)$ is easily expressed in terms of the contours:

$$\sum_{(x,y)} \sigma_x \sigma_y = \text{(no. of bonds in } \theta) \; = \; 2\sum_i^n |\gamma_i| \tag{1.2}$$

so we will describe the probability distribution defined by (1.1) by giving to each member of the ensemble $M^{\boldsymbol{\tau}}(\tau)$ of allowed contour configurations $(\gamma_1, \ldots, \gamma_n)$ the weight

$$w_{\boldsymbol{\tau}}(\gamma_1, \ldots, \gamma_n) \; = \; e^{-\beta \sum_{i=1}^n |\gamma_i|} \tag{1.3}$$

and probability $w_{\boldsymbol{\tau}}(\gamma_1, \ldots, \gamma_n)/Z(M^{\boldsymbol{\tau}}(\theta), \beta)$, where $Z(M^{\boldsymbol{\tau}}(\theta), \beta)$ is the partition function

$$Z(M^{\tau}(\theta), \beta) \stackrel{def}{=} \sum_{(\gamma_1,\dots,\gamma_n)\in M^{\tau}(\theta)} e^{-\beta \sum_{i=1}^{n} |\gamma_i|} \tag{1.4}$$

When the boundary condition is $+1$ or -1 we denote the ensemble by $M^+(\theta)$ or $M^-(\theta)$. More generally, we will also consider other ensembles of configurations and other weights. For any ensemble, M, of configurations in a finite set and any translationally invariant function $\mu(\gamma)$ we define the partition function by:

$$Z(M, \mu) \stackrel{def}{=} \sum_{(\gamma_1,\dots,\gamma_n)\in M} e^{\sum_i |\gamma_i|} \tag{1.5}$$

For example, we will consider the canonical ensembles $M^{\tau}(\theta, m)$ of allowed configurations with a given magnetization $\sum_{x\in\theta} \sigma_x = m|\theta|$.

In Sections 3, 4, 5 and 6 we will consider the situation where we have periodic boundary conditions in the horizontal direction, i.e. instead of the infinite planar lattice $\mathbb{Z}^2$ we consider an infinitely long cylinder $\Omega_{\infty,N}$ with circumference N.

The above considerations are valid for regions $\theta \subset \Omega_{\infty,N}$ as well; one only has to remember that contours can now also go around the cylinder. Especially we are going to consider a cylindrical region $\Omega \subset \Omega_{\infty,N}$ with flat top and bottom and with height N^{δ} for some $\delta > 1$. For it, the four boundary conditions $+$ on the top, $-$ at the bottom etc., will be denoted by $+-$, $-+$, etc.

For any of these boundary conditions the number of "big" countours, i.e. contours going around the cylinder, must have a given parity, even for $++$ and $--$, and odd for $+-$ or $-+$. We will denote by $M_0^{++}(\Omega)$ etc. the ensembles of contours defined by the extra restriction that no contour is allowed to go around $\Omega_{\infty,N}$ or any connected component of the complement of θ, (θ is allowed to have "holes" in it)..

Finally we will also have occasion to consider the ensemble of "c-small" contours in Ω defined by the extra restriction that all closed contours have a length bounded by $c \log |\Omega| = c \log N^{1+\delta}$ for a given c. We will only consider some fixed value of c, e.g. $c = 1/300$. An added subscript c will be used to denote this ensemble. A contour is called "c- large" if it is neither "c-small" nor "big".

Concerning the style used in several of the proofs we remark that the abun- dance of estimates involving numerical constants is not designed to impress the reader with their high accuracy; they are in fact often quite crude. We think however that it is more informative to give values of constants involved instead of writing the customary const., because then it is easier to keep in mind that they do not after all depend on some other parameters, which are later changed. We have not explicitly worked out an estimate of the critical value of β above which our proofs go through, because we have not strived to make the most careful estimates possible at several points anyhow. The phrase "for β large and N large" often used means: for all β above a value independent of N, and all N above a value possibly dependent on β.

Finally we remark that we often freely ignore unimportant rounding off effects due to the fact that the magnetization in a region is an integer $= |\theta$ mod. 2, and we write: magnetization $= m|\theta|$ for any m, height $= N^\delta$ etc. instead of the more cumbersome correct integer expressions.

2 The translationally invariant equilibrium states

In this section we investigate the question of how many phases can coexist in equilibrium. We shall give the following answer to this question: Assume that β is large enough. Then any translationally invariant equilibrium state is a convex linear combination of two pure states describing the up-magnetized and down-magnetized pure phases. In order to make this statement precise we next introduce some definitions. Let Ω be a finite region on the lattice and let $F_\Omega(\boldsymbol{\tau})$ be a given probability distribution over the set of boundary conditions. The corresponding equilibrium state in Ω is then described by the set of correlation functions

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle = \sum_{\boldsymbol{\tau}} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\boldsymbol{\tau}} \, P_\Omega(\boldsymbol{\tau}). \tag{2.1}$$

An equilibrium state of the infinite system is defined as a set of correlation functions $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle$ which can be written, for a suitable choice of the sequence P_Ω as

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle = \lim_{\Omega \to \infty} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{P_\Omega} \tag{2.2}$$

for all $\{x_1, \dots, x_n\}$. We say that it is a translationally invariant equilibrium state if furthermore

$$\langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle = \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle \tag{2.3}$$

for all a. For β large enough it is known that there are at least two different equilibrium states, which will be denoted by $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_+$ and $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_-$. These states are obtained as

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm} = \lim_{\Omega \to \infty} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm,\Omega} \tag{2.4}$$

They describe the up-magnetized and down-magnetized pure phases. We shall next expose some of the special physical properties of these two states, which justify why one uses this terminology. In the following we will need Lemma 2.1 below. From it the existence of the limits Eq.(2.4) follows, with some uniformity in Ω.

Lemma 2.1: *If D is the distance of $\{x_1, \dots, x_n\}$ from the boundary of Ω we have*

$$|\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm} - \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm,\Omega}| \leq f(x_1, \dots, x_n, D) \tag{2.5}$$

where $f(x_1, \dots, x_n, D)$ is a translationally invariant function tending to zero as $D \to \infty$.

The proof of this lemma is given in Appendix 2A. We also show that the two states $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm}$ are translationally invariant, and furthermore that

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_+ = (-1)^n \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_- \tag{2.6}$$

by the obvious symmetry argument. They are also pure phases in the sense of Section 1, *i.e.* they are extremal points of the set of all translationally invariant equilibrium states. We can deduce this from the following cluster property:

$$\lim_{d(x_1 \dots x_n; y_1 \dots y_m)} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm} = \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm} \langle \sigma_{y_1} \dots \sigma_{y_m} \rangle_{\pm} \tag{2.7}$$

A proof of these facts is given in Appendix 2A. Moreover, let us consider the free energy $f(\beta, h)$ and the equilibrium state $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_h$ in an external field $h \neq 0$. It has recently been proved by Ruelle, [7], that this equilibrium state is unique, extremal, translationally invariant and analytic in $h \neq 0$. It has also been proved, [8] that

$$\begin{aligned} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm} &= \lim_{h \to 0^+} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_h \\ \langle \sigma_x \rangle_{\pm} &= \pm m^*, \quad m^* = \frac{\partial f}{\partial h}(\beta, 0^+) \end{aligned} \tag{2.8}$$

Let us now state the main result of this section.

Theorem 2.1: *If $\beta > \log 3$, any translationally invariant equilibrium state $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle$ is given for some $\alpha, 0 \le \alpha \le 1$, by*

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle = \alpha \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_+ + (1-\alpha) \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_- \tag{2.9}$$

Before proving the theorem we make two observations. First, about the value of β above which the theorem holds. It can in fact be proved for $\beta > \log \mu_0$ where μ_0 is the "connective constant" for the self-avoiding walks on the 2-dimensional lattice we consider. It is well known that $\log \mu_1$ for completely self-avoiding walks is an estimate from above of the critical β_b to within 9%. We notice that in order to get the proof of Theorem 2.1 we do not need the techniques which are developed in Section 4. This is the reason why the theorem can be proved so close to critical temperature.

The second observation concerns the value of the spontaneous magnetizat It is an old question whether its value m^* given in Eq.(2.8) coincides with the value

$$m_0 = (1 - (\sinh \beta)^{-4})^{1/8} \tag{2.10}$$

computed from the definition

$$m_0^2 = \lim_{d(x,y) \to \infty} \langle \sigma_x \sigma_y \rangle, \tag{2.11}$$

where $\langle \sigma_x \sigma_y \rangle$ is the limit of the two-spin correlation function with periodic boundary conditions. Since Theorem 2.1 says that $\langle \sigma_x \sigma_y \rangle$ is independent of the boundary conditions, the identity $m_0 = m^*$ for β large enough follows from it. One nee only to apply the last formulae in Eq.(2.8) and use the strong cluster property of $\langle \sigma_x \sigma_y \rangle_+$.

For the proof of Theorem 2.1 it is convenient to introduce the averaged correlation functions. They are defined as:

$$\begin{aligned} \overline{\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle}_{\boldsymbol{\tau}, \Omega} &\stackrel{def}{=} |\Omega|^{-1} \sum_a \langle \sigma_{x_1 + a} \dots \sigma_{x_n + a} \rangle_{\boldsymbol{\tau}, \Omega} \\ \overline{\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle}_{P_\Omega} &\stackrel{def}{=} |\Omega|^{-1} \sum_a \langle \sigma_{x_1 + a} \dots \sigma_{x_n + a} \rangle_{P_\Omega}, \end{aligned} \tag{2.12}$$

where the sum runs over all the a's such that $\{x_1 + a, \dots, x_n + a\} \subset \Omega$. The reason for defining the above averages lies in the fact that if $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle$ is a set of translationally invariant correlation functions verifying Eq.(2.2),(2.3), then one can find a suitable sequence of distributions P_Ω such that for all $\{x_1, \ldots, x_n\}$:

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle = \lim_{\Omega \to \infty} \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{P_\Omega}, \tag{2.13}$$

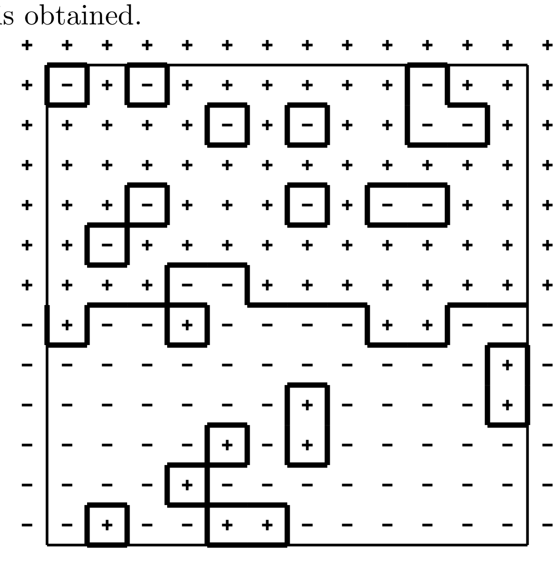
see [2]. This fact, together with the remark that $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{P_\Omega}$ is a convex combination of the $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\boldsymbol{\tau}}$, will imply Theorem 2.1 if the following lemma holds:

Lemma 2.2: *If $\beta > \log 3$ one can find a family of numbers a $\alpha_{\Omega,\boldsymbol{\tau}} \le 1$ and such that*

$$\begin{aligned} &|\overline{\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle}_{\boldsymbol{\tau},\Omega} - \alpha_{\boldsymbol{\tau},\Omega} \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_+ - (1 - \alpha_{\boldsymbol{\tau},\Omega}) \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_- | \\ &= g(x_1 \ldots x_n, \Omega) \end{aligned} \tag{2.14}$$

where $g(x_1 \ldots x_n, \Omega)$ is a translationally invariant and $\boldsymbol{\tau}$-independent function tending to zero as $\Omega \to \infty$.

Let us first describe the physical idea from which the proof of the lemma is obtained.



A boundary condition and the η-lines of a spin configuration, (spins associated with γ-lines are not drawn). Regions $1, 3, 4, 6,$ are positive; $2,$ and $5,$ negative; 2 is not connected.

Fig.3

Let Ω be a given square box containing L^2 points, and let $\boldsymbol{\tau}$ be a fixed boundary condition. For each spin configuration $X \in M^{\boldsymbol{\tau}}(\Omega)$ draw the contours $\gamma_1, \ldots, \gamma_n, \eta_1, \ldots, \eta_s$ associated to X in the way we have described in Section 1. We observe that the open contours $\eta_1, \ldots, \eta_s$ divide the box into $s + 1$ disjoint regions $\theta_1, \ldots, \theta_{s+1}$ which are such that either all spins adjacent to the boundary from the inside are all +1 or all −1. We call the regions "positive" or "negative' according to whether the first or second case happens. See Fig. 3.

Let now $\{x_1, ..., x_n\}$ be a given set of points inside the "big" box Ω, and suppose that $\beta > \log 3$. Then:

(a) The open contours $\eta_1, \ldots, \eta_s$ have possibly a length of the order of L since they must join points on the boundary, but they tend to be not too long, in order to keep the energy small. They are therefore very far from all but a negligille fraction of the translates of $\{x_1, ..., x_n\}$.

(b) A translate $\{x_1 + a, ..., x_n + a\}$ is thus almost always in the "middle" of some θ_i and therefore $\langle \sigma_{x_1+a} \cdots \sigma_{x_n} + a \rangle_{\boldsymbol{\tau},\Omega} \sim \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_+$ if θ_i is a positive region or $\sim \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_-$ if θ_i is a negative region.

Lemma 3.1 gives us the precise statement corresponding to the physical remark 2, whereas we can formulate the physical remark 1 by means of the following lemma, which is also proved in Appendix 2A.

Lemma 2.3 *If* $\beta > \log 3$ *then*

$$P(\boldsymbol{\tau}, L^{\frac{4}{3}}) \overset{def}{=} \text{prob}\{\sum_{i=1}^{s} |\eta_i| \geq L^{\frac{4}{3}}\} \leq \varepsilon(L) \tag{2.15}$$

where $\varepsilon(L)$ *is a function independent of* $\boldsymbol{\tau}$ *and tending to zero as* $L \to \infty$.

Using the notation Eq.(1.3),(1.4) we can compute $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\boldsymbol{\tau},\Omega}$ as

$$\begin{aligned} \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\boldsymbol{\tau},\Omega} = & \sum_{\eta_1 \ldots \eta_s} \sum_{\boldsymbol{\sigma}}^{*} (\sigma_{x_1} \ldots \sigma_{x_n}) \frac{w_{\boldsymbol{\tau}}(\boldsymbol{\sigma})}{Z(M^{\boldsymbol{\tau}}(\Omega), \beta)} \\ & + \varepsilon(x_1 \ldots x_n, \boldsymbol{\tau}), \end{aligned} \tag{2.16}$$

where the first sum runs over all the sets of possible open contours, and the second contours. The function $\varepsilon(x_1 \ldots x_n, \boldsymbol{\tau})$ is according to the above lemma bounded by:

$$|\varepsilon(x_1 \ldots x_n, \boldsymbol{\tau})| \leq \varepsilon(L) \tag{2.17}$$

Suppose that $\{x_1, ..., x_n\} \subset \theta_i$, then

$$\sum_{\boldsymbol{\sigma}}^{*} (\sigma_{x_1} \ldots \sigma_{x_n}) \frac{w_{\boldsymbol{\tau}}(\boldsymbol{\sigma})}{Z(M^{\boldsymbol{\tau}}(\Omega), \beta)} = P_{\boldsymbol{\tau}}(\eta_1 \ldots \eta_s) \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\pm,\theta_i}, \tag{2.18}$$

where the sign has to be chosen to be the same as the one of the region θ_i and $P_{\boldsymbol{\tau}}(h_1, \ldots, \eta_s)$ is the probability of the spin configurations having $\eta_1, \ldots, \eta_s$ as open contours.

This formula follows from the fact that if $\eta_1, \dots, \eta_s$ are fixed then the probabilities of the spin configurations inside the regions η_i are independent and generated by the weights $w_\pm(\boldsymbol{\sigma})$. Let $N_+(\eta_1, \dots, \eta_s)$ be the number of points in the positive θ_i's and put

$$\alpha_{\boldsymbol{\tau},\Omega} = \sum_{\boldsymbol{\sigma}} P_{\boldsymbol{\tau}}(\eta_1 \dots \eta_s) \, \frac{N_+(\eta_1 \dots \eta_s)}{L^2}, \tag{2.19}$$

Denote also by $A(\eta_1 \dots \eta_s)$ the set of points at a distance less than $\frac{1}{2} L^{\frac{1}{3}}$ from the open contours $\eta_1 \dots \eta_s$. Then we find, using (2.16),(2.17),(2.18) and definition Eq.(2.12):

$$\begin{aligned} |\overline{\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle}_{\boldsymbol{\tau},\Omega} - \alpha_{\boldsymbol{\tau},\Omega} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_+ - (1 - \alpha_{\boldsymbol{\tau},\Omega}) \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_- | \\ \leq \varepsilon(L) + 2 f(x_1 \dots x_n, L^{\frac{1}{3}}) + C \frac{L^{\frac{5}{3}}}{L^2}, \end{aligned} \tag{2.20}$$

where the first term comes from the error term in Eq.(2.16), the second comes from the replacement of $\langle \sigma_{x_1+a} \cdots \sigma_{x_n} + a \rangle_{\pm,\theta_i}$ by $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_\pm$ for all the a's such that $x_1 + a, \dots, x_n + a$ does not intersect $A(\eta_1, \dots, \eta_s)$ and from the use of Lemma 3.1 to estimate the error involved. Finally the third term comes from the contribution of the a's such that $x_1 + a, \dots, x_n + a$ intersects $A(\eta_1, \dots, \eta_s)$. The factor $L^{\frac{5}{3}} = 2 \, \frac{1}{2} L^{\frac{1}{3}} \, L^{\frac{4}{3}}$ bounds the number of points in $A(\eta_1, \dots, \eta_s)$, and C is $\boldsymbol{\tau}$-independent and depends only on $\max_{i,j} d(x_i, x_j)$. Formula Eq.(2.20) proves Lemma 2.2.

We finally remark that the validity of Theorem 2.1 is not restricted to the 2-dimensional Ising model. The same techniques and proofs, (with the appropriate notion of contours) easily extend to any number of dimensions. Furthermore, in the 2-dimensional case, as we already mentioned, the condition $\beta > \log 3$ can be replaced by the weaker condition $\beta > \log \mu_0$. To be convinced of these facts one only needs to examine the proof of Lemma 2.3.

Lemma 2.1 and Lemma 2.3 are proved in Appendix 1.

3 Description of the phase separation and definition of the surface tension

In this section we describe in more detail the properties of the phase separation taking place in the ensemble $M^{+-}(\Omega, m)$ described in Section 1 for $m = \alpha m^* + (1-\alpha)(-m^*)$, $0 < \alpha < 1$. We also state the definition of surface tension we use. The proofs of the occurrence of the phase separation and the existence of surface tension are given in Sections 5, and 6.

Consider the cylinder Ω with circumference N and height N^δ, $\delta > 1$, introduced in Section 1, and let the subensemble $\widetilde{M}_0^{+-}(\Omega, m)$ of $M^{+-}(\Omega, m)$ be defined by the restrictions:

(a) There is only one "big" contour, λ, going around the cylinder, and its length is bounded by $(2\log 3)N/\beta$.

(b) The area of the region Ω_λ above λ is restricted by $|\,|\Omega_\lambda| - \alpha|\Omega|\,| \leq a|\Omega|^p$, and hence a similar bound holds for the area below λ.

(c) The total magnetization of the region Ω_λ is restricted by $|M_\lambda - m^*|\Omega_\lambda|\,| \leq a|\Omega|^p$, and hence a similar bound holds for the magnetization below λ.

(d) The total length of the c-large contours is bounded by N/β.

Theorem 3.1: *If β is large enough and $0 < \alpha < 1$, $\delta > 1$, then the probability of $\widetilde{M}_0^{+-}(\Omega, m)$ in $M^{+-}(\Omega, m)$ converges to 1 as $N \to \infty$, i.e.*

$$\lim_{N\to\infty} \frac{Z(\widetilde{M}_0^{+-}(\Omega, m), \beta)}{Z(M^{+-}(\Omega, m), \beta)} = 1 \tag{3.1}$$

for any $a > 0$ and a suitable p, $0 < p < 1$. (Any p satisfying $1 > p$, $p > (1 + c\log 3)/2$, $p > 2/(1+\delta)$, $p > (1 + 1/(1+\delta))/2$ is "suitable").

Theorem 3.1 thus says that a phase separation as described in Section 1 takes place with very high probability in the ensemble $M^{+-}(\Omega, m)$.

This picture of the phase separation is the basis for the following definition of the surface tension between the co-existing phases. A basic property of the partition function of any thermo-dynamic system is the extensivity of its logarithm. *E.g.* $|\Omega|^{-1}|\log Z^{+-}(\Omega, m) \to -\beta f(\beta, m)$ as $|\Omega| \to \infty$, where $f(\beta, m)$ is the limiting free energy per unit volume. Surface effects are manifested in terms proportional to the area of the surface between interacting phases in the difference $\log Z^+ + \beta f(\beta, m)$. For the ensemble $M^{+-}(\Omega, m)$, where there is typically one surface between the two phases and one at each end of the cylinder, one would therefore expect to have an asymptotic relation:

$$\log Z(M_0^{+-}(\Omega, m), \beta) = -\beta f(\beta, m)|\Omega| + \tau N + 2\tau' N + o(N), \tag{3.2}$$

τ being the surface tension between the two phases and τ' that between each phase and the fixed spins. τ is the quantity we want to study and τ' a "spurious" contribution associated directly with the boundary condition.

To extract τ we compare Eq.(3.2) to the corresponding expression expected for an ensemble consisting only of one phase, e.g. $M^{++}(\Omega, m^*)$:

$$\log Z(M_0^{++}(\Omega, m^*), \beta) = -\beta f(\beta, m^*)|\Omega| + 2\tau' N + o(N), \tag{3.3}$$

In fact, in Section refsec5 we are going to see that typically there is not going to be any big contour in this case, and for symmetry reasons the contributions from the bases in Eq.(3.2) and (3.3) should be the same. It is $f(\beta, m^*) = f(\beta, -m^*)$ also, for symmetry reasons, so, since $f(\beta, m)$ should be the sum of the contribution from each phase, $f(\beta, m) = \alpha f(\beta, m^*) + (1-\alpha) f(\beta, -m*) = f(\beta, m*)$. This should allow us to extract τ and define it by the relation

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z(M_0^{+-}(\Omega, m), \beta)}{Z(M^{++}(\Omega, m^*), \beta)} \tag{3.4}$$

Indeed, in Section 6 we prove

Theorem 3.2: *If β is large enough and $0 < \alpha < 1$, $\delta > 1$, then the limit Eq.(3.4) exists and can also be expressed as the limit of a partition function over the possible shapes of the line of separation λ:*

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \sum_{|\lambda| \le N(1+\frac{2\log 3}{\beta})} e^{-\beta|\lambda| + \mu(\lambda, \beta)} \tag{3.5}$$

where $\mu(\lambda, \beta)$ is a certain weight function (defined in Eq.(6.6)); τ is thus independent of α and directly associated with the Line of separation between the phases.

The proof of Section 6 could easily be extended to show also the existence of τ' defined in Eq.(3.3) thereby fully justifying Eq.(3.3) and (3.4), see [6]. The reason why we consider a very long cylinder ($\delta > 1$) is that we can then easily exclude spurious boundary effects, which could occur for α small if α comes near the ends of the cylinder, as it will be seen in the proof. (See also comments in Section 7.)

4 Cluster theory for a pure phase

In this section we are going to study in some detail the ensemble $M_0^+(\theta)$ of configurations defined in Section 1 for a region θ on the infinitely long cylinder $\Omega_{\infty,N}$ or on the planar lattice $\mathbb{Z}^2$. We derive a convenient "virial expansion" of the partition function, which allows us to study its dependence on the region θ, and we also derive estimates for the probability distributions of groups of contours.

We have seen in Section 1 that a configuration $X \in M_0^+(\theta)$ is characterized as a family $(\gamma_1, \ldots, \gamma_n)$ of closed self-avoiding contours in θ, and that its probability is given by:

$$P_{M_0^+(\theta)}(X) = \begin{cases} e^{\sum_{i=1}^n \mu(\gamma_i)}/Z(M_0^+(\theta), \mu) & \text{if } (\gamma_1, \ldots, \gamma_n) \text{ are compatible} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We are of course mainly interested in the special weight function $\mu(\gamma) = e^{-\beta|\gamma|}$, but we also need to consider other weights, so we carry through the discussion for a general translationally invariant weight restricted by $\mu(\gamma) \leq -b\,|\gamma|$ for some constant $b > 0$. For definiteness we consider configurations on an infinite cylinder $\Omega_{\infty,N}$ below, but keep in mind that the arguments are also valid for an infinite planar lattice.

The starting point of our discussion is the observation that the Boltzmann factor $\varphi(\gamma_1, \ldots, \gamma_n)$, can be expressed in terms of a "pair interaction" $f(\gamma, \gamma')$ between the contours as follows:

$$\varphi(\gamma_1, \ldots, \gamma_n) = e^{\sum_{i=1}^n \mu(\gamma_i)} \prod_{i<j} f(\gamma_i, \gamma_j), \tag{4.2}$$

where $f(\gamma, \gamma')$ is defined by

$$f(\gamma, \gamma') = \begin{cases} 1 & \text{if } \gamma, \gamma' \text{ compatible} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Notice that this is only true if we do not allow big contours or contours going around "holes" of θ. We can thus consider the system as a "gas" of contours with "interaction" determined by $f(\gamma, \gamma')$ and "chemical potentials" $\mu(\gamma)$. We can then apply to this "gas" the theory of "low activity" cluster expansions and obtain convenient expressions for the quantities of interest valid for "low activities" $e^{\mu(\gamma)} = e^{-\beta|\gamma|}$, *i.e.* for low temperatures. We use the "algebraic method" to treat the ex- pansion as described in [9, p.86] for an ordinary gas and in [10][4] for a lattice gas.

In order to introduce the "algebraic method" we consider the set of finite configurations $X = (\gamma_1, \ldots, \gamma_n)$ of not big contours on $\Omega_{\infty,N}$. The

[4]The paper however contains a combinatorial error in the last section; the error has been copied in [8] but it is corrected in Section 4 of the present paper. See also [11] or, for an alternative correction, [12].

contours $(\gamma_1, \ldots, \gamma_n)$ are allowed to be incompatible and even to coincide. More formally, a configuration is a function $X(\gamma)$ with non-negative integer values such that $N(X) = \sum_\gamma X(\gamma) < \infty$, and $X(\gamma)$ is the "multiplicity" of γ in X. Configurations can be added in the obvious sense: $(X_1 + X_2)(\gamma) = X_1(\gamma) + X_2(\gamma)$.

We also consider the space F of real-valued functions of configurations $\varphi(X)$ such that $\sup_{N(X)=n} |\varphi(X)| < \infty$ for all n, and its subspaces F_0 and F_1 of functions such that $\varphi(\emptyset = 0$ and $\varphi(\emptyset) = 1$ respectively. If $\varphi_1, \varphi_2 \in F$ we can define their convolution product by:

$$(\varphi_1 * \varphi_2)(X) = \sum_{X_1+X_2=X} \varphi_1(X_1)\varphi_2(X_2). \tag{4.4}$$

The sum is finite since X is finite and $\varphi_1 * \varphi_2 \in F$ also. Next we define the exponential, corresponding to the convolution, for $\varphi \in F_0$:

$$\begin{aligned}(\operatorname{Exp}\varphi)(X) =& \sum_{n\geq 0} \frac{1}{n!}\varphi^{n*}(X) \\ =& 1(X) + \sum_{n\geq 1} \frac{1}{n!} \sum_{X_1+\ldots+X_n=X} \varphi(X_1)\cdots\varphi(X_n).\end{aligned} \tag{4.5}$$

$\varphi^{0*}(X) \overset{def}{=} 1(X)$ is defined to be $\mathbf{1}$ if $X = \emptyset$ and 0 otherwise. For each X the sum in Eq.(4.5) is finite because $\varphi \in F_0$, and $\operatorname{Exp}\varphi \in F_1$. We also define the corresponding logarithmic function for $\varphi \in F_1$ as follows. If $\varphi = \mathbf{1} + \varphi_0$ with $\varphi_0 \in F_0$ then:

$$\begin{aligned}(\operatorname{Log}\varphi)(X) =& \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\varphi_0^{n*}(X) \\ =& \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{X_1+\ldots+X_n=X} \varphi_0(X_1)\cdots\varphi_0(X_n).\end{aligned} \tag{4.6}$$

Again, each sum is finite and $\operatorname{Log}\varphi \in F_0$. We also see that $\operatorname{Log}\operatorname{Exp}\varphi_0 = \varphi_0$ for $\varphi_0 \in F_1$ and $\operatorname{Exp}\operatorname{Log}\varphi_1 = \varphi_1$ for $\varphi_1 \in F_1$.

The main reason for introducing this convolution product is, as we are going to see, the following product property. If $\chi(X)$ is a character function in the sense that $\chi(X + 1 + X_2) = \chi(X_1)\chi(X_2)$, *i.e.* if $\chi(X) = \prod_\gamma z(\gamma)^{X(\gamma)}$ for some function $z(\gamma)$, (which relation we write as $\chi(X) = z^X$) and if $\sum_X |\varphi_i(X)z^X < \infty$, $i = 1, 2$, then

$$\sum_X (\varphi_1 * \varphi_2)(X) z^X = (\sum_X (\varphi_1)(X) z^X)\,(\sum_X (\varphi_2)(X) z^X) \tag{4.7}$$

and

$$\sum_X (\operatorname{Exp}\varphi)(X) z^X = \exp\,(\sum_X \varphi(X) z^X) \tag{4.8}$$

if $\varphi \in F_0$.

Especially we are going to use Eq.(4.8) when

$$z(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ lies in some regon } \theta \\ & \text{and does not go around its “holes”} \\ 0 & \text{otherwise} \end{cases}$$

so that we get

$$\sum_{X \subset \theta} (\operatorname{Exp}\varphi)(X) z^X = \exp\,(\sum_{X\subset\theta} \varphi(X) z^X) \tag{4.9}$$

($X \subset \theta$ means that all the contours of X lie in θ).

We also introduce a”derivation’ operator D_X on F defined by:

$$(D_X\varphi)(Y) = \varphi(X+Y)\frac{(X+Y)!}{Y!} \tag{4.10}$$

with $X! = \prod_\gamma X(\gamma)!$. In terms of it the following ”Taylor’s formula” is valid:

$$\sum_X \varphi(X)(u+v)^X = \sum_X \frac{u^X}{X!}\sum_\gamma (D_X\varphi)(Y)v^Y,$$

from which the following rules can be proved

$$\begin{aligned}
&D_\gamma(\varphi_1 * \varphi_2) = (D_\delta\varphi_1) * \varphi_2 + \varphi_1 * (D_\gamma\varphi_2)\\
&\frac{D_X(\varphi_1 * \varphi_2)}{X!} = \sum_{X_1+X_2} (\frac{D_{X_1}\varphi_1}{X_1!})(\frac{D_{X_2}\varphi_2}{X_2!})\\
&D_\gamma(\operatorname{Exp}\varphi) = (D_\gamma\varphi) * (\operatorname{Exp}\varphi)\\
&\frac{D_X(\operatorname{Exp}\varphi)}{X!} = \Big(\sum_{n\ge1}\frac{1}{n!}\sum_{X_1+\dots X_n = X,\, X_i\neq 0}(\frac{D_{X_1}\varphi}{X_1!}) * (\frac{D_{X_n}\varphi}{X_n!})\Big) * (\operatorname{Exp}\varphi)
\end{aligned} \tag{4.11}$$

We now use the notions above to analyze the Boltzmann factor $\varphi \in F_1$ de- fined in Eq.(4.2). Let $\varphi^T \in F_O$ be defined by $\varphi^T \overset{def}{=} \operatorname{Log}\varphi$, so that $\varphi = \operatorname{Exp}\varphi^T$. Then

$$Z(M_0^+(\theta),\mu) = \sum_{X\subset\theta} \varphi(X) = \exp(\sum_{X\subset\theta} \varphi^T(X)) \tag{4.12}$$

so we see that the logarithm of the partition function can be expanded in terms of φ^T. The important feature of this expansion is that $\varphi^T(X)$ does not depend on the region θ and can be estimated in a suitable way, and this will allow us to study how the partition function depends on θ.

We can obtain a "graphological" formula for $\varphi^T(X)$ like the one for the Ursell functions used in the theory of the Mayer expansion as follows.

If we define $\widetilde{\varphi}^T(\gamma_1,\ldots,\gamma_n) = \varphi^T(X)\,X!$ for all the $\frac{n!}{X!}$ ordered sequences $(\gamma_1,\ldots,\gamma_n) = X$ we can write

$$\begin{aligned}\sum_X \varphi^T(X)z^X &= \sum_{n\ge0}\sum_{n(X)=n} z^X\varphi^T(X) \sum_{\substack{\gamma_1,\ldots,\gamma_n\\(\gamma_1,\ldots,\gamma_n)=X}} \frac{X!}{n!}\\ &= \sum \frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n} z(\gamma_1)\cdots z(\gamma_n)\widetilde{\varphi}^T(\gamma_1,\ldots,\gamma_n),\end{aligned} \tag{4.13}$$

which expression is used e.g. in [ll]. In terms of $\widetilde{\varphi}^T$ the expression for the convolution can be found by considering

$$\begin{aligned}&(\sum_X(\varphi_1^T)(X_1)z^{X_1})\,(\sum_{X_2}(\varphi_2^T)(X_2)z^{X_2})\\ &= \sum_{n_1,n_2\ge0}\frac{1}{n_1!n_2!}\sum_{\substack{\gamma'_1,\ldots,\gamma'_{n_1}\\ \gamma''_1,\ldots,\gamma''_{n_2}}} z(\gamma'_1)\cdots z(\gamma''_{n_2})\widetilde{\varphi}_1^T(\gamma'_1,\ldots,\gamma'_{n_1})\widetilde{\varphi}_2^T(\gamma''_1,\ldots,\gamma''_{n_2})\\ &= \sum_{n\ge0}\frac{1}{n!}\sum_{n_1+n_2=n}\frac{n!}{n_1!n_2!}\sum_{\gamma_1,\ldots,\gamma_n} z(\gamma_1)\cdots z(\gamma_n)\\ &\qquad\cdot\widetilde{\varphi}_1^T(\gamma_1,\ldots,\gamma_{n_1})\widetilde{\varphi}_2^T(\gamma_{n_1+1},\ldots,\gamma_n)\end{aligned}$$

But for any partition of $N = \{1,\ldots,n\}$ into $N_1\cup N_2$ with $|N_1| = n_1,, N_2 = n_2$ the last sum can be written

$$\sum_{\gamma_1,\ldots,\gamma_n} z(\gamma_1)\cdots z(\gamma_n)\,\widetilde{\varphi}_1^T(\gamma_i;\,i\in N_1)\,\widetilde{\varphi}_2^T(\gamma_i;\,i\in N_2)$$

by a suitable change of dummy variables. Because there are $\frac{n!}{n_1!\,n_2!}$ such partitions the sum with n_1,n_2 given can be written:

$$\sum_{\gamma_1,\ldots,\gamma_n} z(\gamma_1)\cdots z(\gamma_n) \sum_{\substack{N_1\cup N_2=N\\ |N_i|=n_i}} \widetilde{\varphi}_1^T(\gamma_i;\, i\in N_1)\,\widetilde{\varphi}_2^T(\gamma_i;\, i\in N_2),$$

and we get

$$\begin{aligned}(\sum_X(\varphi_1^T)(X_1)z^{X_1})\,(\sum_{X_2}(\varphi_2^T)(X_2)z^{X_2}) &= \sum_X z^X(\varphi_1*\varphi_2)(X)\\ =\sum_{n\geq 0}\frac{1}{n!}\sum_{\gamma_1,\ldots,\gamma_n} z(\gamma_1)\cdots z(\gamma_n)(\widetilde{\varphi}_1*\widetilde{\varphi}_2)(\gamma_1,\ldots,\gamma_n)\end{aligned} \tag{4.14}$$

with

$$(\widetilde{\varphi}_1*\widetilde{\varphi}_2)(\gamma_1,\ldots,\gamma_n) = \sum_{N_1\cup N_2=N} \widetilde{\varphi}_1^T(\gamma_i;\, i\in N_1)\,\widetilde{\varphi}_2^T(\gamma_i;\, i\in N_2), \tag{4.15}$$

This function being symmetric in $\gamma_1,\ldots,\gamma_n$ we can conclude that $(\widetilde{\varphi_1^T * \varphi_2^T}) = \widetilde{\varphi}_1^T*\widetilde{\varphi}_2^T$ and by induction that $\widetilde{(\varphi^T)^{n*}} = (\widetilde{\varphi}^T)^{n*}$ for all n. This means that $(\operatorname{Exp}\varphi^T)(X)$ is also given by the expression:

$$\begin{aligned}\varphi(X) =&(\operatorname{Exp}\varphi^T)(X) = \frac{1}{X!}\sum_{m=0}^{\infty}\frac{\widetilde{\varphi}^{T\,m*}(\gamma_1,\ldots,\gamma_n)}{m!}\\ =&\frac{1}{X!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{N_1\cup\ldots\cup N_m=N}\widetilde{\varphi}^T(\gamma_i;\, i\in N_1)\cdots\widetilde{\varphi}^T(\gamma_i;\, i\in N_m)\\ =&\frac{1}{X!}\sum_{m=0}^{\infty}{}'\sum_{N_1\cup\ldots\cup N_m=N}\widetilde{\varphi}^T(\gamma_i;\, i\in N_1)\cdots\widetilde{\varphi}^T(\gamma_i;\, i\in N_m),\end{aligned} \tag{4.16}$$

where the last sum is over all different partitions of N into any number of parts. This formula is useful for finding φ^T when φ is defined in terms of a pair interaction as in Eq.(4.2) as we will now see.

In Eq.(4.2) write $f(\gamma,\gamma') = 1 + g(\gamma,\gamma')$ with

$$g(\gamma,\gamma')\begin{cases}0 & \text{if } \gamma,\gamma' \text{ compatible}\\ -1 & \text{if } \gamma,\gamma' \text{ incompatible}\end{cases} \tag{4.17}$$

and expand the product: ($\varphi = \widetilde{\varphi}$ because $\varphi(X) = 0$ when $X! \neq 1$)

$$\begin{aligned}\widetilde{\varphi}(\gamma_1,\ldots,\gamma_n) =& e^{\sum_i^n \mu(\gamma_i)} \prod_{1\le i<j\le n} (1+\gamma(\gamma_i,\gamma_j)) \\ =& e^{\sum_i^n \mu(\gamma_i)} \sum_G \prod_{\{i,j\}\in G} \gamma(\gamma_i,\gamma_j)\end{aligned} \tag{4.18}$$

The last summation is over all subgraphs G on N. Each such G induces a partition of N into connected components and isolated points, so if we define $g(M)$ for any $M \subseteq N$ by:

$$g(M) = \begin{cases} e^{\sum_M \mu(\gamma_i)} \sum_{G_M} \prod_{\{i,j\}\in G_M} g(\gamma_i,\gamma_j) & \text{if } |M| \ge 2 \\ e^{\mu(\gamma_i)} & \text{if } M = \{i\} \\ 0 & \text{if } M = \emptyset \end{cases} \tag{4.19}$$

where the sum is over all connected graphs on M, we realize that

$$\widetilde{\varphi}(\gamma_1,\ldots,\gamma_n) = \sum_{N_1\cup\ldots\cup N_m = N}{}' g(N_1)\ldots g(N_n), \tag{4.20}$$

and we see from Eq.(4.16) that $g(M) = \widetilde{\varphi}^T(\{\gamma_i\}_{i\in M})$ for $M \subseteq N$. We thus finally get the following formula for $\varphi^T(X) = \varphi^T(\gamma_1,\ldots,\gamma_n)$: construct the graph G with vertices $\{1,...,n\}$ and edges $\{i,j\}$ corresponding to incompatible pairs $\{\gamma_i,\gamma_j\}$. Then

$$\varphi^T(\gamma_1,\ldots,\gamma_n) = \frac{1}{X!} e^{-\sum_{i=1}^n \mu(\gamma_i)} \sum_{C\subseteq G} (-1)^{\#\text{ of edges in } G}, \tag{4.21}$$

where the sum is over all connected subgraphs of G visiting all the points $\{1,...,n\}$. From this expression we see that $\varphi^T(\gamma_1,\ldots,\gamma_n) = 0$ if G is not connected, *i.e.* if $(\gamma_1,\ldots,\gamma_n)$ can be split into two groups such that every γ in one is compatible with every γ in the other, which fact will be used repeatedly in the following.

We also see that φ^T is transitionally invariant, and that if $(\gamma_1,\ldots,\gamma_n)$ is a configuration on the cylinder $\Omega_{\infty,N}$ which can be drawn on the infinite planar lattice as well, without changing the "compatibilities", then φ^T is the same for the two configurations. This happens e.g. if $(\gamma_1,\ldots,\gamma_n)$ does not "encircle" $\Omega_{\infty,N}$.

We next derive a "Kirkwood-Salsburg" equation like that used in [10], which will allow us to get convenient estimates for φ^T and the correlation functions $\rho_\theta(X)$ defined by: $\rho_\theta(X) = P(\text{the contours in } X \text{ are present})$

$$\begin{aligned}\rho_\theta(X) =&\text{Probability}(\{\text{contours in } X \text{ are present}\})\\ =&\frac{\sum_{Y\subset\theta}\varphi(X+Y)}{\sum_{Y\subset\theta}\varphi(Y)} = \sum(\varphi^{-1} * D_X\varphi)(Y)\end{aligned} \tag{4.22}$$

(φ^{-1} defined by $\varphi^{-1} * \varphi = 1$ well defined if $\varphi(\emptyset) \neq 0$, and $\varphi(X) = 0$ if $X! \neq 1$). Define $\Delta_X(Y)$ by

$$\Delta_X(Y) = (\varphi^{-1} * D_X\varphi)(Y) = \sum_{Y_1+Y_2=Y} \varphi^{-1}(Y_1)\varphi(X+Y_2) \tag{4.23}$$

Then a recursive equation for $\Delta_X(Y)$ can be derived as follows. From Eq.(4.23) we see that $\Delta_X(Y) = 0$ if X is not a compatible set. Consider $\Delta_{\gamma+X}(Y)$ for $\gamma + X$ compatible. From EQ.(4.2) follows that

$$\begin{aligned}\varphi(\gamma+X+Y_2) =&e^{\mu(\gamma)}\varphi(X+Y_2)\prod_{\gamma'\in Y_2}(1+g(\gamma,\gamma'))\\ =&e^{\mu(\gamma)}\varphi(X+Y_2)\sum_{S\subseteq Y_2}^{*}(-1)^{N(S)}\end{aligned}$$

if Y_2 is without "multiplicities", and the sum is over sets S, all of whose elements are incompatible with γ. We then get:

$$\begin{aligned}\Delta_{\gamma+X}(Y) =& \sum_{Y_1+Y_2=Y}\varphi^{-1}(Y_1)\varphi(\gamma+X+Y_2)\\ =&e^{\mu(\gamma)}\sum_{Y_1+Y_2=Y}\varphi^{-1}(Y_1)\varphi(X+Y_2)\sum_{S\subseteq Y_2}^{*}(-1)^{N(S)},\end{aligned} \tag{4.24}$$

because only Y_2's without multiplicities contribute. Put $Y_2 = S + Y_3$. Then

$$\begin{aligned}\Delta_{\gamma+X}(Y) =&e^{\mu(\gamma)}\sum_{S\subseteq Y}^{*}(-1)^{N(s)}\sum_{Y_1+Y_3=Y-S}\varphi^{-1}(Y_1)\varphi(X+S+Y_3)\\ =&e^{\mu(\gamma)}\sum_{S\subseteq Y}^{*}\Delta_(S+X)(Y-S).\end{aligned} \tag{4.25}$$

In the sum $S = \emptyset$ is to be included, and $\Delta_\emptyset(Y) = \mathbf{1}(Y)$. Observe that this equation determines $\Delta_X(Y)$ with $N(X) + N(Y) = m + 1$ in terms of $\Delta_X(Y)$ with $N(X) + N(Y) = m$ for $m = 0, 1, \ldots$, successively. This makes it possible to derive the following useful estimate. Let I_m be defined by:

$$I_m = \sup_{\substack{\gamma_1,\dots,\gamma_n \\ m\ge n\ge 1}} \sum_{Y,\,N(Y)=m-n} |\Delta_{\gamma_1,\dots,\gamma_n}(Y)|\, e^{\frac{b}{2}\sum_i |\gamma_i|}. \tag{4.26}$$

We can then deduce from Eq.(4.25):

$$\begin{aligned}
&\sum_{Y,\,N(Y)+N(X)=m} |\Delta_{\gamma+X}(Y)| e^{\frac{b}{2}(|\gamma|+|X|)} \\
&\le \sum_{Y,\,N(Y)+N(X)=m} \sum^{*}_{S\subseteq Y} |\Delta_{X+S}(Y-S)| e^{\frac{b}{2}(|Y|+|X|)-b|\gamma|} \\
&\le \sum^{*}_{S} I_m e^{-\frac{b}{2}(|\gamma|+|S|)} e^{-\frac{b}{2}|\gamma|} \sum_{n\ge 0} \frac{1}{n!} (\sum_{\sigma\cap\gamma\neq\emptyset} e^{-\frac{b}{2}|\sigma|})^n \\
&\le I_m\, e^{-\frac{b}{2}|\gamma|} \exp(\sum_{p\in\gamma}\sum_{\sigma\ni p} e^{-\frac{b}{2}|\sigma|}) \le I_m e^{-\frac{b}{2}|\gamma|} \exp(|\gamma| \sum_{\ell=4}^{\infty} (3e^{-\frac{b}{2}})^\ell) \\
&\le I_m \exp|\gamma|(-\frac{b}{2} + 3^4 e^{-2b}(1-3e^{-\frac{b}{2}}))
\end{aligned} \tag{4.27}$$

if $3e^{-\frac{1}{2}b} < 1$.(Here and at several other instances we use the fact that the number of different contours of length ℓ that go through a given point is less than 3^ℓ). If $3e^{-\frac{1}{2}\beta} \le \frac{1}{2}$ it is easy to see that the last expression in Eq.(4.27) is $\le I_m e^{-1.8\,b}$ because $|\gamma| \ge 4$. We can conclude that

$$I_{m+1} \le I_m e^{-(1.8)m} \qquad \text{for } m \ge 1 \tag{4.28}$$

if $3e^{-\frac{1}{2}b} < \frac{1}{2}$. Because:

$$\begin{aligned}
I_1 =& \sup_{\gamma} |\Delta_\gamma(\emptyset)| e^{\frac{b}{2}|\gamma|} = \sup_{\gamma} |(\varphi^{-1} * D_\gamma)(\emptyset) e^{\frac{b}{2}|\gamma|} \\
=& \sup_{\gamma} |\varphi(\gamma)| e^{\frac{b}{2}|\gamma|} \le \sup_{|\gamma|\ge 4} e^{-\frac{b}{2}|\gamma|} = e^{-2b}
\end{aligned} \tag{4.29}$$

we see from Eq.(4.28) that:

$$I_m \le e^{-(1.8)m} \tag{4.30}$$

if $3e^{-\frac{1}{2}b} < \frac{1}{2}$. This bound allows us to estimate φ^T as follows. From Eq.(4.11) we see that

$$|\Delta_\gamma(X) = (\varphi^{-1} * D_\gamma\varphi)(X) = D_\gamma\varphi^T(X) = \varphi^T(\gamma+X)\frac{(\gamma+X)!}{X!} \tag{4.31}$$

and we can derive a bound for the quantity $\sum_X |\varphi^T(\gamma+X)|$, which will be very useful:

$$\begin{aligned}\sum_X |\varphi^T(\gamma+X)| &\le \sum_{m=1}^{\infty} \sum_{N(X)=m-1} |\Delta_\gamma(X)| \le \sum_{m=1}^{\infty} e^{-\frac{1}{2}b|\gamma|-(1.8)m\,b} \\ &\le \frac{e^{-\frac{1}{2}b|\gamma|-(1.8)m\,b}}{1-e^{-(1.8)b}} \le (1.1)e^{-\frac{1}{2}b|\gamma|-(1.8)\,b}\end{aligned} \tag{4.32}$$

if $3e^{-\frac{1}{2}\beta} < \frac{1}{2}$. We can also bound $\sum_{X\ni p} |\varphi^T(X)|$ for any given point p:

$$\begin{aligned}&\sum_{X\ni p} |\varphi^T(X)| \le \sum_{\gamma\ni p}\sum_X |\varphi^T(\gamma+X)| \le (1.1)\sum_{\gamma\ni p} e^{-\frac{1}{2}b|\gamma|-(1.8)\,b} \\ &\le (1.1)e^{(1.8)b}\sum_{\ell=4}^{\infty}(3e^{-\frac{1}{2}b})^\ell \le (2.2)e^{-(3.8)b}\end{aligned} \tag{4.33}$$

if $3e^{-\frac{1}{2}\beta} < \frac{1}{2}$.

These bounds also allow us to estimate $\sum_{X\ni p, X\,i\,Q} |\varphi^T(X)$, where p is a point and Q any set of points, and $X\,i\,Q$ means that X intersects Q. Let d be the distance between p and Q, and divide the above sum into two parts according to whether $N(X) \ge d^{\frac{1}{2}}$ or $N(X) < d^{\frac{1}{2}}$. The first part can be estimated as in Eq.(4.32) using Eq.(4.30):

$$\begin{aligned}\text{first part} &\le \sum_{\gamma\ni p}\sum_{N(X)\ge\sqrt{d}} |\varphi^T(\gamma+X) \le \sum_{\gamma\ni p} e^{-\frac{1}{2}b|\gamma|}\sum_{m\ge d} e^{-(1.8)mb} \\ &\le (1.1)e^{-(1.8)b\sqrt{d}}\sum_{\ell=4}^{\infty}(3e^{-\frac{1}{2}b})^\ell \le (2.2)e^{-2b-(1.8)\,b\,\sqrt{d}}\end{aligned} \tag{4.34}$$

If $N(X) < d^{\frac{1}{2}}$ we can conclude that the longest contour in X, $\overline{\gamma}$, has a length $\ell \ge d^{\frac{1}{2}}$ if $\varphi^T(X) \ne 0$. This is true because if $\varphi^T(X) \ne 0$ we know that the contours in X form one overlapping group, so that

$$\le \text{ length of } X \;\le\; \ell\, N(X) \le \ell d^{\frac{1}{2}}, \text{ and } \ell \ge d^{\frac{1}{2}}$$

We also have $d(p,\overline{\gamma}) \le \ell N(X) \le \ell d^{\frac{1}{2}}$ for the same reason, so $\overline{\gamma}$ must intersect the square with side $2\ell d^{\frac{1}{2}}$ centered at p. The second part can therefore be estimated as follows using Eq.(4.32):

$$
\begin{aligned}
\text{second part } &\leq \sum_{\ell \geq \sqrt{d}} \sum_{\substack{|\widetilde{\gamma}|=\ell \\ d(p,\widetilde{\gamma}) \leq \ell\sqrt{d}}} \sum_{X} |\varphi^T(\widetilde{\gamma}+X)| \\
&\leq \sum_{\ell \leq \sqrt{d}} \sum_{\substack{|\widetilde{\gamma}|=\ell \\ d(p,\widetilde{\gamma}) \leq \ell\sqrt{d}}} (1.1) e^{-(1.8)b - \frac{1}{2}b|\widetilde{\gamma}|} \\
&\leq (1.1)e^{-(1.8)b} \sum_{\ell \geq \sqrt{d}} (4\ell^2 d)(3e^{-\frac{1}{2}b})^{\ell} \leq 60 e^{-(1.8)b} d^2 (3e^{-\frac{1}{2}b})^{\sqrt{d}}
\end{aligned}
\tag{4.35}
$$

if $3e^{-\frac{1}{2}\beta} < \frac{1}{2}$, and we finally find that

$$
\sum_{X \ni p,\, X\, i\, q} |\varphi^T(X)| \leq 60 e^{-(1.8)b} (d^2+1)(3e^{-\frac{1}{2}b})^{\sqrt{d}} d(p,Q) \tag{4.36}
$$

if $3e^{-\frac{1}{2}b} < \frac{1}{2}$.

(*Remark in proof*: Eq.(4.36) can actually be improved so that $d(p,Q)$ occurs in the exponent instead of $d^{\frac{1}{2}}(p,Q)$. This is easily seen because from Eq.(4.21) it follows that if $b > b_0$ and $2e^{-\frac{1}{2}b_0} \leq \frac{1}{2}$ then $\varphi^T(\gamma_1,\ldots,\gamma_n) = e^{\sum_i (\mu(\gamma_i)+b_0|\gamma_i|)} \varphi_0^T(\gamma_1,\ldots,\gamma_n)$ with φ_0^T defined by the weight $\mu_0(\gamma) = -b_0|\gamma|$. The φ_0^T can be estimated by Eq.(4.36), so the corresponding estimate for φ^T can be improved by a factor $e^{-(b-b_0)d(p,Q)}$. However we do not need to use this sharper bound.)

For the proof of Lemma 5.4 in Appendix 2 we need an estimate of the quadratic form

$$
Q = \sum_{\gamma_1,\gamma_2 \subset \theta} \Delta\mu(\gamma_1) \cdot \frac{\partial^2 \log Z(M^+_{0,c}(\theta),\beta)}{\partial\mu(\gamma_1)\partial\mu(\gamma_2)} \cdot \Delta\mu(\gamma_2)
$$

which we now derive. (Remember that $M^+_{0,c}(\theta)$ is the sub-ensemble of $M^+_0(\theta)$ having only c-small contours.) Notice that the partition function for this ensemble can also be expressed in terms of φ^T as in Eq.(4.9). We only redefine $z(\gamma)$ as:

$$
\begin{cases} z(\gamma) & \text{if } \gamma \text{ is } c\text{-small} \\ 0 & \text{otherwise} \end{cases} \tag{4.37}
$$

and get

$$Z(M^+_{0,c}(\theta),\beta) = \sum_X \varphi(X) z_c^X = \exp \sum_X \varphi^T(X) z_c^X. \tag{4.38}$$

We can thus use the same expansions as before only adding the restriction on the length in the summations (which we denote by $\sum_{X\subset\theta}^c$). From the definition of $\varphi(X)$ in Eq.(4.2) we see that

$$\begin{aligned} \frac{\partial\varphi(X)}{\partial\mu(\gamma)} = \frac{\partial^2\varphi(X)}{\partial\mu(\gamma)^2} &= \begin{cases} \varphi(X) & \text{if } X = \gamma + Y \text{ for some } Y \\ 0 & \text{otherwise} \end{cases} \\ \frac{\partial^2\varphi(X)}{\partial\mu(\gamma_1)\partial\mu(\gamma_2)} &= \begin{cases} \varphi(X) & \text{if } X = \gamma_1 + \gamma_2 + Y \text{ for some } Y \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.39}$$

if $\mu_1 \neq \mu_2$, so that we get, remembering the definition of the correlation functions in Eq. (4.2):

$$\begin{aligned} Q =& \sum_{\gamma\subset\theta}^c [\rho_{\theta,c}(\gamma) - \rho^2_{\theta,c}(\gamma)] \\ &+ \sum_{\substack{\gamma_1,\gamma_2\subset\theta \\ \gamma_1\neq\gamma_2}}^c \Delta\mu(\gamma_1)\,[\rho_{\theta,c}(\gamma_1,\gamma_2) - \rho_{\theta,c}(\gamma_1)\rho_{\theta,c}(\gamma_2)]. \end{aligned} \tag{4.40}$$

From Eq.(4.22) we get the important bound

$$\rho_{\theta,c}(\gamma) \leq e^{-\beta|\gamma|} \tag{4.41}$$

because all terms in the numerator contain this factor, and the remaining sum is contained in the denominator. (The same bound and argument is also true for $\pi_{0,c}(\gamma)$, the probability that γ is an outer contour). Hence the first term, Q', in Q can directly be bounded by

$$Q' \leq \sum_{\gamma\subset\theta}^c (\Delta\mu(\gamma))^2 e^{-b|\gamma|} \leq |\theta| \sum_{(\gamma)}^c (\Delta\mu(\gamma))^2 e^{-b|\gamma|} \tag{4.42}$$

if $\Delta\mu(\gamma)$ is translationally invariant. To bound the second term, $Q^\sim$, we need to estimate the clustering property of $\rho_{0,c}(\gamma_1,\gamma_2)$ as $d(\gamma_1,\gamma_2) \to \infty$. From Eq.(4.22) follows that

$$\begin{aligned} &\rho_{\theta,c}(\gamma_1,\gamma_2) - \rho_{\theta,c}(\gamma_1)\rho_{\theta,c}(\gamma_2) = \\ &\sum_Y z_c^Y \Delta_{\gamma_1,\gamma_2}(Y) - (\sum_Y z_c^Y \Delta_{\gamma_1}(Y))(\sum_Y z_c^Y \Delta_{\gamma_2}(Y)) \\ &= \sum_Y z_c^Y (\Delta_{\gamma_1,\gamma_2}(Y) - (\Delta_{\gamma_1} * \Delta_{\gamma_2})(Y)), \end{aligned} \tag{4.43}$$

The last term can be expressed in terms of φ^T, because as in Eq.(4.31) we have:

$$D_{\gamma_1}\varphi^T = \varphi^{-1} * D_{\gamma_1}\varphi = \Delta_{\gamma_1} \tag{4.44}$$

so that (using Eq.(4.11))

$$D_{\gamma_2\gamma_2}\varphi^T = \varphi^{-1} * D_{\gamma_1\gamma_2}\varphi - \varphi^{-2} * D_{\gamma_1}\varphi * D_{\gamma_2}\varphi = \Delta_{\gamma_1\gamma_2} - \Delta_{\gamma_1} * \Delta_{\gamma_2}. \tag{4.45}$$

To estimate $\sum_{Y\subset\theta}^{c} |D_{\gamma_1\gamma_2}\varphi^T(Y)|$ in terms of $d(\gamma_1,\gamma_2)$ we proceed as in the derivation of Eq.(4.36), and then we need to estimate also $D_{\gamma_1\gamma_2\gamma_3}\varphi^T(Y)$. It can be expressed in terms of $\Delta_X(Y)$ if we differentiate Eq.(4.45) once more: ($\gamma_1,\gamma_2,\gamma_3$ are all different)

$$\begin{aligned} D_{\gamma_1\gamma_2\gamma_3}\varphi^T &= \varphi^{-1} * D_{\gamma_1\gamma_2\gamma_3}\varphi - \varphi^{-2} * \Big(D_{\gamma_1}\varphi * D_{\gamma_2\gamma_3}\varphi \\ &+ D_{\gamma_2}\varphi * D_{\gamma_1\gamma_3}\varphi + D_{\gamma_3}\varphi * D_{\gamma_1\gamma_2}\varphi\Big) + 2\varphi^{-3} * D_{\gamma_1}\varphi * D_{\gamma_2}\varphi * D_{\gamma_3}\varphi \\ &= \Delta_{\gamma_1\gamma_2\gamma_3} - \Delta_{\gamma_1} * \Delta_{\gamma_2\gamma_3} - \Delta_{\gamma_2} * \Delta_{\gamma_1\gamma_3} - \Delta_{\gamma_3}\Delta_{\gamma_1\gamma_2} + 2\Delta_{\gamma_1} * \Delta_{\gamma_2} * \Delta_{\gamma_3} \end{aligned} \tag{4.46}$$

Any terms appearing in Eq.(4.45),(4.46),(4.47) can be estimated using Eq. (4.30), which says:

$$\sum_{\substack{Y\\ N(Y)=m-N(X)}} |\Delta_X(Y)| \le e^{-\frac{1}{2}b|X|-1.8mb} \tag{4.47}$$

We thus get for a typical term

$$\begin{aligned} &\sum_{Y,N(Y)=m-\sum N(X_i)} |(\Delta_{X_1} * \ldots * \Delta_{X_n})(Y)| \\ &\le \sum_{Y,N(Y)=m-\sum N(X_i)} \sum_{Y_1+\ldots+Y_n=Y} |\Delta_{X_1}(Y_1)|\cdots|\Delta_{X_n}(Y_n)| \\ &\le \sum_{m_1+\ldots+m_n=m} \sum_{\substack{N(Y_i)=m_i-N(X_i)\\ i-1,\ldots n}} |\Delta_{X_1}(Y_1)|\cdots|\Delta_{X_n}(Y_n)| \\ &\le \sum_{m_1+\ldots+m_n=m} e^{-\frac{1}{2}b\sum N(X_i)|X_i| - 1.8mb} \le m^n e^{-\frac{1}{2}b\sum N(X_i)|X_i| - 1.8mb} \end{aligned} \tag{4.48}$$

and hence:

$$\sum_{N(Y)=m-2} |D_{\gamma_1\gamma_2}\varphi^T(Y)| \leq 2m^2 e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)-1.8mb} \tag{4.49}$$

$$\sum_{N(Y)=m-3} |D_{\gamma_1\gamma_2\gamma_3}\varphi^T(Y)| \leq 6m^3 e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|+|\gamma_3|)-1.8mb} \tag{4.50}$$

Now we can turn to the quantities in Eq.(4.43), ($d \stackrel{def}{=} d(\gamma_1, \gamma_2)$):

$$\begin{aligned} |\rho_{\theta,c}(\gamma_1,\gamma_2) - \rho_{\theta,c}(\gamma_1)\rho_{\theta,c}(\gamma_2)| &\leq \sum_Y |D_{\gamma_1\gamma_2}\varphi^T(Y)| \\ &\leq \sum_{N(Y)\geq d^{\frac{1}{2}}} \cdot + \sum_{N(Y)<d^{\frac{1}{2}}} \cdot \end{aligned} \tag{4.51}$$

The first sum is bounded using Eq.(4.49):

$$\sum_{N(Y)\geq d^{\frac{1}{2}}} \cdot \leq 2e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} \sum_{d^{1/2}+2}^{\infty} m^2 e^{-1.8mb} \tag{4.52}$$

For the second we use the agument leading to Eq.(4.35): the longest member of Y, $\widetilde{\gamma}$, must have $|\widetilde{\gamma}| \geq d^{\frac{1}{2}}$ and $d(\gamma_1, \widetilde{\gamma}) \leq |\widetilde{\gamma}| d^{\frac{1}{2}}$, ($|\gamma_1| \leq |\gamma_2|$) say,

$$\begin{aligned} \sum_{N(Y)<d^{\frac{1}{2}}} \cdot &\leq \sum_{\ell \geq d^{\frac{1}{2}}} \sum_{|\widetilde{\gamma}|=\ell,\, d(\gamma_1,\widetilde{\gamma})\leq \ell d^{\frac{1}{2}}} \sum_Y |D_{\gamma_1\gamma_2\widetilde{\gamma}}\varphi^T(Y)| \\ &\leq \sum_{\ell\geq d^{\frac{1}{2}}} 4\ell^2 d|\gamma_1| \sum_{|\widetilde{\gamma}|=\ell,\gamma\ni O} 6\, e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|+|\widetilde{\gamma}|)} \sum_1^{\infty} m^3 e^{-1.8mb} \\ &\leq 24d|\gamma_1| (\sum_1^{\infty} m^3 e^{-1.8\,m\,b}) (\sum_{\ell} \geq d^{\frac{1}{2}} \ell^2 (3e^{-\frac{1}{2}b})^{\ell}) e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} \end{aligned} \tag{4.53}$$

To continue it is convenient to have a simple estimate of sums of the tpe $S_{N,p} = \sum_{n=N}^{\infty} n^p a^n$ with $0 < a < 1$,

Lemma 4.1: $S_{N,p} \leq \frac{a^N p!(1+p\,N^p)}{(1-a)^{p+1}}$ *for any integers* $N, p \geq 0$.

Proof. The mean value theorem tells us that $(n+1)^p - n^p \leq p\,((n+1)^p)$, for $n, p \geq 0$, hence $\sum_{n=N-1}^{\infty} p(n+1)^{p-1} a^{n+1}$ for $N \geq 1$ and $S_{N,p} - aS_{N,p} - (N-1)^p a^N \leq pS_{N,p-1}$, so we have the recursion:

$$
\begin{aligned}
S_{N,p} \leq & \frac{pS_{N,p-1}}{(1-a)} + \frac{(N-1)^p a^N}{1-a} \qquad \text{for } p = 0, 1, \ldots, \; N = 1, 2, \ldots, \\
S_{N,0} = & \frac{a^N}{1-1}, \qquad \text{for } N \geq 0
\end{aligned}
\tag{4.54}
$$

it gives

$$
\begin{aligned}
S_{N,p} \leq & \frac{a^N p!}{(1-a)^{p+1}} \sum_0^p \frac{(N-1)^q (1-a)^q}{q!} \\
\leq & \frac{a^B p!(1+p(N-1)^p)}{(1-a)^{p+1}} \leq \frac{a^N p!(1+p\,N^p)}{(1-a)^{p+1}}
\end{aligned}
\tag{4.55}
$$

for $p \geq 0, N \geq 1$. For $p \geq 1, N = 0$ we have

$$
S_{0,p} = S_{1,p} \leq \frac{ap!}{(1-a)^{p+1}} \leq \frac{a^0 p!(1+0^p)}{(1-a)^{p+1}},
$$

and for $p = 0$: $S_{N,0} = \frac{a^N}{1-a} = \frac{a^N 0!(1+0N^0)}{(1-a)}$, so the inequality is true for all $N, p \geq 0$.

It is now easy to bound the series in Eq.(4.52) and (4.53) if $3e^{-\frac{1}{2}b} \leq \frac{1}{2}$ which implies that $e^{-(1.8)\,b} < \frac{1}{500}$.

$$
\sum_{N(X) \geq d^{\frac{1}{2}}} \cdot \leq (0.2) e^{-b} (1+d)(500)^{-d^{\frac{1}{2}}} e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} \tag{4.56}
$$

$$
\sum_{N(X) \geq d^{\frac{1}{2}}} \cdot \leq 1600 e^{-b} d^{\frac{1}{2}} 2^{-d^{\frac{1}{2}}} |\gamma_1| e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} \tag{4.57}
$$

and we get

$$
\begin{aligned}
& |\rho_{\theta,c}(\gamma_1,\gamma_2) - \rho_{\theta,c}(\gamma_1)\rho_{\theta,c}(\gamma_2)| \\
& \leq 1600\, e^{-b} (|\gamma_1| g_2|)^{\frac{1}{2}} e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} (1 + d(\gamma_1,\gamma_2)^2) 2^{-d(\gamma_1,\gamma_2)^{\frac{1}{2}}}
\end{aligned}
\tag{4.58}
$$

With the help of Eq.(4.58) we can estimate $Q\tilde{}$ and get:

$$
\begin{aligned}
Q^{\sim} \leq & \sum_{\gamma_1,\gamma_2\subset\theta}{}^{c}(1600)e^{-b}|\Delta\mu(\gamma_1)||\Delta\mu(\gamma_2)|(|\gamma_1|g_2|)^{\frac{1}{2}}e^{-\frac{1}{2}b\,(|\gamma_1|+|\gamma_2|)} \\
& \cdot(1+d(\gamma_1,\gamma_2)^2)2^{-d(\gamma_1,\gamma_2)^{\frac{1}{2}}} \\
\leq & (1600)e^{-b}\sum_{\gamma_1\subset\theta}{}^{c}|\Delta\mu(\gamma_1)||\gamma_1|^{\frac{1}{2}}e^{-\frac{1}{2}b\,|\gamma_1|}\sum_{(\gamma_2)}{}^{c}|\Delta\mu(\gamma_2)||\gamma_2|^{\frac{1}{2}}e^{-\frac{1}{2}b\,|\gamma_2|} \\
& \cdot\sum_{\gamma_2\in(\gamma_2)}(1+d(\gamma_1,\gamma_2)^2)2^{-d(\gamma_1,\gamma_2)^{\frac{1}{2}}}
\end{aligned}
\tag{4.59}
$$

The innermost sum is bounded by:

$$
\begin{aligned}
& |\gamma_1||\gamma_2|\sum_1^{\infty}(4d)(d^2+1)2^{-d^{\frac{1}{2}}} \leq |\gamma_1||\gamma_2|\sum_1^{\infty}4n^2(n^4+1)2n2^{-(n-1)} \\
& \leq |\gamma_1||\gamma_2|\,2.1\,10^7,
\end{aligned}
\tag{4.60}
$$

so we get

$$
\begin{aligned}
Q^{\sim} & \leq(3.5)10^{10}e^{-b}|\theta|(\sum_{(\gamma)}{}^{c}|\Delta\mu(\gamma)|\,|\gamma|^{\frac{3}{2}}e^{-\frac{1}{2}b|\gamma|})^2 \\
& \leq(3.5)10^{10}e^{-b}|\theta|(\sum_{(\gamma)}{}^{c}|\Delta\mu(\gamma)|^2)(\sum_{(\gamma)}|\gamma|^3e^{-\frac{1}{2}b|\gamma|}).
\end{aligned}
$$

The last sum is bounded by E Z32-Z C 100, so, remembering the estimate 0 finally get:

$$
Q \leq 4\,10^{12}\,e^{-b}\,|\theta|\,(\sum_{(\gamma)}|\Delta\mu(\gamma)|^2e^{-\frac{1}{2}b|\gamma|}), \tag{4.61}
$$

which is the estimate needed. (Actually the restriction on the length of the contours was not used in the proof, so a similar estimate is valid in the unrestricted ensemble).

5 The phase separation

In this section we give a proof of the phase separation as described in Theorem 3.1 using the method of proof used by Minlos-Sinai adapted to our situation. The proof depends on several estimates of various probabilities, which we formulate as a series of lemmas. Their proofs are given in detail in [4, 5], and will not be repeated here except Lemma 5.4, which is proved in Appendix 2.

The first lemma says that in all the ensembles of interest we need only consider the "minimal" ensembles having a minimal number of big, contours, because their probabilities converge to 1:

Lemma 5.1: *The following inequalities hold*

$$\lim_{N\to\infty} \frac{Z(M_0^{++}(\Omega),\beta)}{Z(M^{++}(\Omega),\beta)} = 1 \tag{5.1}$$

$$\lim_{N\to\infty} \frac{Z(M_0^{+-}(\Omega),\beta)}{Z(M^{+-}(\Omega),\beta)} = 1 \tag{5.2}$$

$$\lim_{N\to\infty} \frac{Z(M_0^{+-}(\Omega,m),\beta)}{Z(M^{+-}(\Omega,m),\beta)} = 1 \tag{5.3}$$

if β is large and $m = (2\alpha-1)m^*$ with $0 < \alpha < 1$.

To prove that some set $E \subset M_0^{+-}(\Omega,m)$ has a small probability in the "difficult" ensemble $M_0^{+-}(\Omega,m)$ the following argument will allow us to conider the "simpler" ensemble $M_0^{+-}(\Omega)$ instead:

$$\frac{Z(E,\beta)}{Z(M_0^{+-}(\Omega,m),\beta)} = \frac{Z(E,\beta)}{Z(M_0^{+-}(\Omega),\beta)} \frac{Z(M_0^{+-}(\Omega),\beta)}{Z(M_0^{+-}(\Omega,m),\beta)}, \tag{5.4}$$

so if we have an upper bound on the last ratio we can conclude that the left hand side goes to zero if the probability of E in $M_0^{+-}(\Omega)$ goes to zero fast enough.

Lemma 5.2: *The following inequality holds*

$$\frac{Z(M_0^{+-}(\Omega),\beta)}{Z(M_0^{+-}(\Omega,m),\beta)} = D(\alpha,\beta)\, N^{2\delta+3} e^{N\delta'(\beta)} \tag{5.5}$$

for some constants $D(\alpha,\beta), \delta'(\beta)$ if β is large and $m = (2\alpha-1)m^*$, and $\delta(\beta)$ goes to zero exponentially as $\beta \to \infty$.

"Fast enough" in the above argument is thus *e.g.* $P_{M_0^{+-}(\Omega)}(E) \le e^{-N^{1+\varepsilon}}$ for some $\varepsilon > 0$. We also need some bound on the length of the big contour λ present in $M_0^{+-}(\Omega,m)$ and of the c-large contours:

Lemma 5.3: *In $M_0^{+-}(\Omega,m)$ the probability that $|\lambda| - N \le \beta^{-1}(2\log 3)N$ and the total length of the c-large contours is less than $N\beta^{-1}$ tends to 1 as $N \to \infty$ for β large and $m = (2\alpha-1)m^*$.*

We finally need the following important estimate of the fluctuations of the total magnetization $M(X)$ in the ensemble $M_{O,c}(\theta)$ of c-small contours in any large subregion θ of Ω:

Lemma 5.4: *Let $\theta \subset \Omega$ be a region such that $|\theta| \geq k|\Omega|$ and $|\partial\theta| \leq k|\theta|^{\frac{1}{2}}$ for some $k > 0$. Then*

$$P_{M^+_{0,c}(\theta)}(|M(X) - m^*| \geq t|\theta|^p) \leq e^{\frac{-t^2|\theta|^{2p-1}}{400\,\delta(\beta)}} \tag{5.6}$$

with $\delta(\beta) = 10^{14}e^{-\beta}$ if t and p are restricted by:

$$\frac{1}{2}(1 + c\log 3) < p < 1, \qquad 2\delta(\beta)^{\frac{1}{2}} \leq t \leq |\theta|^{\frac{1-p}{2}}$$

and if $|\theta|$ and β are large, e.g. if $3e^{-\frac{1}{2}\beta} < \frac{1}{2}$ and $3e^{-\beta} < e^{-\frac{3}{2}c}$.

The lemma thus says that the probability of "large" fluctuations (large than $const\,|\theta|^{\frac{1}{2}}$ have a bound $\exp\frac{(t|\theta|^p)^2}{const\,|\theta|}$ as one would expect for "normal" fluctuations. (θ is allowed to have "holes" in it, but no contour of a configuration in $M^+_{0,c}(\theta)$ encircles any of them.)

With the help of these lemmas we can give a proof of the phase separation along the following lines. Consider first the fluctuations in the area of the region Ω_λ above λ, the big contour of $X \in M_0^{+-}(\Omega, m)$.

We can assume that the bounds of Lemma 5.3 are satisfied by X. If $|\Omega_\lambda|$ deviates much from $\alpha|\Omega|$, *e.g.* if $|\Omega_\lambda| \geq \alpha|\Omega| + a|\Omega|^p$, then either above or below λ the total magnetization M_λ or $m|\Omega| - M_\lambda$ will deviate much from the "expected" values $m^*|\Omega_\lambda|$ ISi,I or $-m^*(|\Omega| - |\Omega_\lambda|)$. Because of the bound on the length of the c-large contours this deviation cannot come from the region enclosed by them or by c-small contours which enclose large contours. Hence it comes from the regions formed only by c-small contours and its probability can be effectively estimated using Lemma 5.4 and shown to be "small enough" on the scale of Lemma 5.2.

The above argument can be made precise as follows. Let E be the subset of $M_0^{+-}(\Omega, m)$ defined by the restrictions $|\Omega_\lambda| \geq \alpha|\Omega| + a|\Omega|^p$, $|\lambda| - N \leq (2\log 3)N\beta^{-1}$ and $|c-\text{large contours} - N\beta^{-1}$. For any configuration $X \in E$ let $\gamma_1, \ldots, \gamma_n$ be the c-large outer contours (if any are present in X) and $\gamma'_1, \ldots, \gamma'_{n'}$ those c-small contours that enclose a c-large contour (if any are present in X) and let A be the area enclosed by all of them. Because a contour has at least the length 4 we see that $4(n+n') \leq |c-\text{large contours}| \leq N\beta^{-1}$ and $A < \frac{1}{16}(|\gamma_1| + \ldots + |\gamma_n|)^2 + \frac{1}{16}n'(c\log|\Omega|)^2 \leq \frac{1}{16}((N\beta^{-1})^2 + N(4\beta)^{-1})$ if N is large and ν not too small.

The magnetization inside $\Gamma = (\gamma_1, \dots, \gamma'_{n'})$, M_0, is thus also bounded by $N^2 = |\Omega|^{\frac{2}{1+\delta}}$, so if $p > \frac{2}{(1+\delta)}$ it is much smaller than $|\Omega|^p$. Let $|\theta_1|$ and $|\theta_2|$ be the regions outside Γ above and below λ and let M_1 and M_2 be their magnetizations respectively. Let also A be split into $A_1 + A_2$ by λ. Because $M_0 + M_1 + M_2 = m|\Omega| = (2\alpha - 1)m^*|\Omega|$ and $A + |\theta_1| + |\theta_2|$ we have

$$\begin{aligned}((M_1 - m^*|\theta_1|) + (M_2 - m^*|\theta_2|)) = (2\alpha - 1)m^*|\Omega| - M_0 - m^*|\theta_1| \\ + m^*(|\Omega| - |\theta_1| - A) = 2m^*(\alpha|\Omega| - |\Omega_\lambda|) + m^*A_1 - m^*A_2 - M_0\end{aligned} \tag{5.7}$$

Consider now the two subsets of E defined by

$$E_1 \,:\, |M_1 - m^*|\theta_1|| \geq m^* a\,|\Omega|^p \qquad \text{and} \qquad E_2 \,:\, |M_1 - m^*|\theta_1|| < m^* a|\Omega|^p.$$

From (5.7) we see that E_2 implies that

$$M_2 + m^*|\theta_2| \leq -m^*(a|\Omega| - |\Omega_\lambda|) + 2N^2, \tag{5.8}$$

which implies that

$$M_2 + m^*|\theta_2| \leq -m^* a|\Omega|^p + 2N^2, \tag{5.9}$$

and also because $M_2 \geq -|\theta_2|$:

$$\begin{aligned}-|\theta_2| + m^*|\theta_2| &\leq m^*(|\Omega| - |\Omega_\lambda| - |\Omega|(1-\alpha)) + 2N^2 \\ &= m^*(|\theta_2| + A^2 - |\Omega|(1-\alpha)) + 2N^2,\end{aligned} \tag{5.10}$$

so $|\theta_2| \geq |\Omega| m^*(1-\alpha) - 3N^2 \geq k|\Omega|$ for some $k > 0$ in this case. These considerations allow us to estimate the probability of E in $M_0^{+-}(\Omega)$ as follows:

$$\begin{aligned}P(E) =& P(E_1) + P(E_2) = \sum_{\lambda,\Gamma} \Big(P(E_1 \,|\, \lambda, \Gamma) + P(E_2 \,|\, \lambda, \Gamma)\Big)\, p(\lambda, \Gamma) \\ &\leq \sum_{\lambda,\Gamma} (P(|M_1 - m^*|\theta_1| \geq m^* a|\Omega|^p \,|\, \lambda, \Gamma) \\ &+ P(|M_2 + m^*|\theta_2| \geq m^* a|\Omega|^p \,|\, \lambda, \Gamma))\, P(\lambda, \Gamma)\end{aligned} \tag{5.11}$$

But when λ and Γ are fixed these last probabilities are computed in the ensembles $M_{0,c}^{+}(\theta_1)$ and $M_{0,c}^{+}(\theta_2)$ respectively, because θ_l and θ_2 only contain c-small contours. Moreover, the conditions on θ_1 and θ_2 for the use of lemma 5.4 are fulfilled (uniformly in λ, Γ), so we get:

$$P_{M_0^{+-}(\Omega)}(E) \leq \delta\, e^{-const\,|\Omega|^{2p-1}} \tag{5.12}$$

We thus see using the bound of Lemma 5.2 that $P_{M_0^{+-}}(\Omega, m)(E) \to 0$ also, if $|\Omega|^{2p-1} = N^{(1+\delta)(2p-1)} > N$, *i.e.* if $p > \frac{1}{2}(1 + \frac{1}{1+\delta})$. Using Lemma 5.3 we then finally see that $P_{M_0^{+-}}(|\Omega_\lambda| \geq \alpha|\Omega| + a|\Omega|^p) \to 0$ also. The case $|\Omega_\lambda| \leq \alpha|\Omega| - a|\Omega|^p$ is treated in the same way.

The fluctuations of the magnetization above and below λ are estimated quite analogously. Consider M_λ the magnetization of Ω_λ, *e.g.*, and define E this time by the restrictions: $X \in M_0^{+-}(\Omega, m), |M_\lambda - m^*|\Omega_\lambda|| \geq a|\Omega|^p$, $|\Omega_\lambda| > k|\Omega|$, $|\Omega| - |\Omega_\lambda| > k|\Omega|$ for a suitable $k > 0$, $|\lambda| - N \leq (2\log 3)N\beta^{-1}$ and $|c$-large contours$| \leq N\beta^{-1}$.

Because as before $|M_\lambda - M_1| \leq N^2$, $|\Omega_\lambda| - |\theta_1| \leq N^2$ we have $|M_1 - m^*|\theta_1| \geq a|\Omega|^p - 2N^2$ in E, and $P(E\,|\,\lambda, \Gamma) \leq P(M_1 - m^*|\theta_1|| \geq a|\Omega|^p - 2N^2\,|\lambda, \Gamma) \leq 3e^{-const\,|\Omega|^{2p-1}}$ by Lemma 5.4, so that $P_{M_0^{+-}(\Omega,m)}(E) \to 0$ for the same p-values as above. From what was just proved we know that $P_{M_0^{+-}(\Omega,m)}(|\Omega_\lambda| > k|\Omega|,\ |\Omega| - |\Omega_\lambda| > k|\Omega|) \to 1$ for a suitable $k > 0$, however, so we know that

$$P_{M_0^{+-}(\Omega,m)}(|M_1 - m^*|\Omega_\lambda|| \geq a|\Omega|^p) \to 0 \tag{5.13}$$

The same argument applies to the magnetization below λ, and Theorem 3.1 is proved.

Lemma 5.4 is proved in Appendix 2.

6 The surface tension

In this section we show that the definition of the surface tension given in Section 3 is allowed in the sense that the limit Eq.(3.4) exists and is independent of α. In the course of the proof the partition functions appearing in Eq.(3.4) will be approximated by simpler objects, and in the end the surface tension will appear as the thermodynamic limit of a partition function of the ensemble of big contours λ, each one having a weight of the form $e^{-\beta|\lambda| + \mu(\lambda,\beta)}$ with $|\mu(\lambda, \beta)| \leq (2.2)|\lambda|^{(3.8)\beta}$. We need yet another estimate similar to Lemma 5.2 which is also proved in [6].

Lemma 6.1: *Let $\theta \subset \Omega$ be a cylinder whose bases are not necessarily flat but are restricted by the condition that their lengths do not exceed $2N$ each and their distance is at least kN^δ for some $k > 0$. Then if $0 \leq m^* - m \leq AN|\theta|^{-1}$ some $A > 0$ we have:*

$$1 \geq \frac{Z(M_0^+(\theta,m),\beta)}{Z(M_0^+(\theta,\beta))} \geq D(\beta)|\theta|^{-\frac{1}{2}} e^{-d(\beta)N^{\frac{1}{2}}} \tag{6.1}$$

for some constants $D(\beta), d(\beta)$ *if* β *is large.*

The significance of this lemma for us will be that the logarithms of the two partition functions differ by an amount which is small compared to a "surface term" τN.

Consider now the formula Eq.(3.4). Lemma 5.1 and 6.1 show that we can replace $Z(M^{++}(\Omega, m*),\beta)$ and $Z(M^{+-}(\Omega,m),\beta)$ by $Z(M^{++}(\Omega),\beta)$ and $Z(M^{+-}(\Omega,m),\beta)$ without changing τ. Furthermore, we consider the ensemble $\widetilde{M}_0^{+-}(\Omega,m)$ of Theorem 3.1 and get:

$$\begin{aligned} Z(\widetilde{M}_0^{+-}(\Omega,m),\beta) \leq \sum_{\lambda}^{\sim} e^{-\beta|\lambda|} \sum_{m^+,m^-} Z(M_0^{++}(\Omega_\lambda, m^+),\beta) \\ \cdot Z(M_0^{--}(\Omega'_\lambda, m^-),\beta) \leq Z(M_0^{+-}(\Omega_\lambda, m^+),\beta) \end{aligned} \tag{6.2}$$

where $\sum_\lambda^\sim$ denotes the sum over the allowed λ's, and m^+, m^- the magnetizations of the regions Ω_λ, Ω'_λ above and below λ are restricted by $m^+|\Omega|_\lambda + m^-|\Omega'_\lambda = m|\Omega|$. By Theorem 3.1 the ratio of the left and right term in Eq.(6.2) tends to 1 as $N \to \infty$, so we can replace $Z_0^{+-}(M_0^{+-}(O,m),\beta)$ by the middle term $\widetilde{Z}$ in the definition of τ.

We now obtain a lower bound on $\widetilde{Z}$ by restricting the sum further. For each Λ in $\sum_\lambda^\sim$, we can find another one λ' congruent to Λ by shifting it vertically until $0 \leq |\Omega_{\prime l'} - \alpha|\Omega| < 2N$ because when λ is shifted one step $|\Omega_\lambda|$ changes at most by $|\lambda| \leq N(1+2\beta^{-1}\log 3) < 2N$ when β is not too small. We denote by $\sum_{(\lambda)}^\sim$ the sum obtained by picking on such translate, λ', for each shape (λ) restricted by $|\lambda| \leq N(1+2\beta^{-1}\log 3)$. We then only pick one term in the sum $\sum_{m^+,m^-}$, namely that having $m^- = -m^*$. It has $m^+|\Omega_\lambda| = m|\Omega| + m^*(|\Omega| - |\Omega_\lambda|) = (2\alpha-1)m^*|\Omega| + m^*|\Omega| - m^*|\Omega_\lambda|$, so $0 \leq (m^* - m^+)|\Omega_\lambda| = 2m^*(|\Omega_\lambda| - \alpha|\Omega|) \leq 4N$, and both $Z(M_0^{++}(O_l, m^+),\beta)$ and $Z(M_0^{--}(O'_l, m^-),\beta)$ can be estimated by Lemma 6.1:

$$\begin{aligned} \widetilde{Z} &\geq \sum_{(\lambda)}^{\sim} e^{-\beta|\gamma|} Z(M_0^{++}(\Omega_\lambda, m^+),\beta) Z(M_0^{--}(\Omega'_\lambda, m^-),\beta) \\ &\geq \frac{D(\beta)^2 e^{-d(\beta)N^{\frac{1}{2}}}}{2\sqrt{\alpha(1-\alpha)}|\Omega|} \sum_{(\lambda)}^{\sim} e^{-\beta|\lambda|} Z(M_0^{++}(\Omega_\lambda),\beta) Z(M_0^{--}(\Omega'_\lambda),\beta) \end{aligned} \tag{6.3}$$

$\widetilde{Z}$ can on the other hand be estimated by:

$$\begin{aligned}&\sum_{\lambda} e^{-\beta|\lambda|} Z(M_0^{++}(\Omega_\lambda),\beta) Z(M_0^{--}(\Omega'_\lambda),\beta)\\ &\le \widetilde{\sum_{\lambda}} e^{-\beta|\lambda|} \widetilde{\sum_{\lambda'\in(\lambda)}} Z(M_0^{++}(\Omega_{\lambda'}),\beta) Z(M_0^{--}(\Omega'_{\lambda'}),\beta)\end{aligned} \tag{6.4}$$

The last sum is over all allowed λ' vertically congruent to (λ). Because of the restrictions defining '$\widetilde{M}_0^{+-}(\Omega, m)$ the distance from any such λ' to the bases of Ω is larger than kN^δ for some $k > 0$. This fact will allow us to find an expression for the product of the two partition functions independent of the position of λ'. Using the expression Eq.(4.12) for the partition function in terms of $\varphi^T(X)$ we get:

$$\begin{aligned}&Z(M_0^{++}(\Omega_{\lambda'}),\beta) Z(M_0^{--}(\Omega'_{\lambda'}),\beta) = \exp(\sum_{X\subset\Omega_{\lambda'}} \varphi^T(X) + \sum_{X\subset\Omega'_{\lambda'}} \varphi^T(X))\\ &= Z(M_0^{++}(\Omega),\beta) \exp(-\sum_{\substack{X\subset\Omega_{\infty,N}\\ X\,i\,\lambda'}} \varphi^T(X) + \sum_{\substack{X\,i\,\lambda'\\ X\,i\,\partial\Omega}} \varphi^T(X)),\end{aligned} \tag{6.5}$$

where $X\,i\,\lambda'$ means that X intersects λ'. The last sum in Eq.(6.5) can be estimated using Eq.(4.36):

$$\sum_{\substack{X\,i\,\lambda'\\ X\,i\,\partial\Omega}} |\varphi^T(X)| \le (\frac{3}{4})^{k^{\frac12}N^{\frac52}} \le (\frac{3}{4})^{N^{\frac12}} \quad \text{if } 3e^{\frac12\beta} \le \frac{1}{2}$$

and N is large uniformly in λ'. Putting

$$\mu(\lambda,\beta) = \sum_{\substack{X\,i\,\lambda\\ X\subset\Omega_{\infty,N}}} \varphi^T(X) \tag{6.6}$$

we have thus obtained the two bounds:

$$\begin{aligned}&\frac{D(\beta)^2 e^{-\left(\frac34\right)^{N^{\frac12}} - d(\beta)N^{\frac12}}}{2\sqrt{\alpha(1-\alpha)}|\Omega|} Z(M_0^{++}(\Omega),\beta) \widetilde{\sum_{(\lambda)}} e^{-\beta|\lambda| - \mu(\lambda,\beta)}\\ &\le\ \widetilde{Z}\ \le\ N^\delta e^{\left(\frac34\right)^{N^{\frac12}}} Z(M_0^{++}(\Omega),\beta) \widetilde{\sum_{(\lambda)}} e^{-\beta|\lambda| - \mu(\lambda,\beta)}\end{aligned} \tag{6.6a}$$

which show that the expression for τ can also be taken to be:

$$\tau = \lim_{N\omega\infty} N^{-1} \log \sum_{(\lambda)}^{\sim} e^{-\beta|\lambda|-\mu(\lambda,\beta)}. \tag{6.7}$$

From Eq.(4.3) follows that

$$|\mu(\lambda,\beta)| \le |\lambda| 2.2 e^{-3.8\beta} \le \varepsilon(\beta)|\lambda| \tag{6.8}$$

if $3e^{-\frac{1}{2}\beta} \le \frac{1}{2}$, so the crude bounds

$$\sum_{\ell=N}^{\infty} e^{-(\beta+\varepsilon(\beta))\ell} \le \sum_{\lambda}^{\sim} \cdot \le \sum_{\ell=N}^{\infty} e^{-(\beta-\varepsilon(\beta))\ell} \tag{6.9}$$

show that

$$-\beta - \varepsilon(\beta) \le \tau \le -\beta + \varepsilon(\beta) + \log 3 \tag{6.10}$$

Put $\sum_{(\lambda)}^{\sim} e^{-\beta|\lambda|-\mu(\lambda,\beta)} = S_N(\beta)$. To show that the limit Eq.(6.7) exists we will use the well known subadditivity argument for $\log S_N(\beta)$ and show that $S_N(\beta)S_M(\beta) \le S_{N+M}(\beta)$ at least approximatively. To this end we note the following properties of the paths appearing in any of the sums, $S_N(\beta)$ *e.g.* let β be not too small, so that $(2log3)\beta^{-1} < \frac{1}{2}$ and hence $|\lambda| - N < \frac{N}{2}$ for all terms in $S_N(\beta)$. For any λ, we can then find a column C on the cylinder with the following properties:

(a) The strip of width one immediately to the right of C only contains one horizontal step of X.

(b) The strip of width $2N^{\frac{1}{3}}$ centered at C contains a portion of $|\lambda|$ at most $N^{\frac{1}{2}}$ long. (This can be seen as follows. The number of horizontal steps of λ which are simple in the sense that there are no other horizontal steps above or below it is at least $\frac{1}{2}N$, because for each group of "multiple" steps at least one unit of the excess length $|\lambda| - N$ is "consumed". Consider for each simple step the strip of width $2N^{\frac{1}{3}}$ centered at its left end, and let L be the shortest length of λ contained in any of these strips. Let M be the maximal size of a family of *disjoint* such strips. Then any simple step has a horizontal distance at most $N^{\frac{1}{3}}$ from this maximal family, so if the maximal strips are widened to $4N^{\frac{1}{3}}$ their union will contain all simple steps. Hence $M \cdot 4N^{\frac{1}{3}} \ge$ the width of the union $\ge \frac{1}{2}N$. Moreover, $LM \le$ the length in the maximal family $\le \frac{3}{2}N$, so $L \le \frac{3N}{2M} \le 12N^{\frac{1}{3}} < N^{\frac{1}{2}}$ if N is large, and we can take C as the column containing the left end of any simple step whose strip contains the length L.)

Using property (a) we can now construct a mapping F which associates to any pair λ_N, λ_M coming from $S_N(\beta)$ and $S_M(\beta)$ a λ_{N+M} included in $S_{N+M}(\beta)$ as follows. "Open up" λ_N and λ_M at some C_N and C_M as described above and join them together on a cylinder with circumference $N+M$ to a closed path λ_{N+M} (first λ_N and then λ_M *e.g.* starting from a fixed origin). λ_{N+M} will be allowed in $S_{N+M}(\beta)$ because of (a). Moreover, at most NM pairs can be mapped on the same $\lambda_{N+M}(\beta)$. Because of restriction (b) $\mu(\lambda, \beta)$ will be nearly additive in this process:

$$\mu(\lambda_N, \beta) = \sum_{\substack{X\, i\, \lambda_N \\ X \subset \Omega_{\infty,N}}} \varphi^T(X) = \sum_{\substack{X\, i\, \lambda_N \\ X\, \not{i} C_N,\ C \subset \Omega_\infty}} \varphi^T(X) \sum_{\substack{X\, i\, \lambda_N \\ X\, i\, C_N,\ C \subset \Omega_\infty}} \varphi^T(X) \quad (6.11)$$

The last sum can be bounded by considering those X that intersect λ_N around C_N and the others separately using Eq.(4.33) and Eq.(4.36).

$$\sum_{\substack{X\, i\, \lambda_N \\ X\, i\, C_N,\ C \subset \Omega_\infty}} |\varphi^T(X)| \leq 2.2\,(N^{\frac{1}{2}} + (\frac{3}{4})^{N^{\frac{1}{6}}}) \quad (6.12)$$

e.g. if $3e^{-\frac{1}{2}\beta} \leq \frac{1}{2}$ and N is not too small. A similar estimate is valid for $\mu(\lambda_M, \beta)$, and for $\mu(\lambda_{N+M}, \beta)$ we get:

$$\begin{aligned}\mu(\lambda_{N+M}, \beta) = & \sum_{\substack{X \text{ betw. } C_N \text{ and } C_M \\ X\, i\, \lambda_{N+M}, X \subset \Omega_{\infty,N+M}}} \varphi^T(X) + \sum_{\substack{X \text{ betw. } C_M \text{ and } C_N \\ X\, i\, \lambda_{N+M}, X \subset \Omega_{\infty,N+M}}} \varphi^T(X) \\ & + \sum_{\substack{X\, i\, C_M \text{ or} \\ X\, i\, \lambda_{N+M}, X \subset \Omega_{\infty,N+M}}} \varphi^T(X)\end{aligned} \quad (6.13)$$

with a similar estimate for the last term. The first two terms in Eq.(6.13) are equal to the corresponding terms in $\mu(\lambda_N, \beta)$ and $\mu(\lambda_M, \beta)$, because $\varphi^T(X)$ does not depend on the cylinder unless X encircles it as explained after Eq.(4.21). We thus see that

$$|\mu(\lambda_{N+M}, \beta) - \mu(\lambda_N, \beta) - \mu(\lambda_M, \beta)| \leq 5\,(N^{\frac{1}{2}} + M^{\frac{1}{2}}) \quad (6.14)$$

e.g. uniformly in (λ_N, λ_M).

These considerations give us the desired approximative subadditivity of $\log S_N(\beta)$ as follows:

$$S_{N+M} = \sum_{(\lambda_{N+M})}^{\sim} e^{-\beta|\lambda_{N+M}|-\mu(\lambda_{N+M},\beta)} \geq \sum_{\substack{(\lambda_{N+M})\\ \varepsilon \text{ range } F}} e^{-\beta|\lambda_{N+M}|-\mu(\lambda_{N+M},\beta)}$$

$$\geq (NM)^{-1} \sum_{(\lambda_N),(\lambda_M)}^{\sim} e^{-\beta(|\lambda_N|+|\lambda_M|)-\mu(\lambda_N,\beta)-\mu(\lambda_M,\beta)-5\,(N^{\frac{1}{2}}+M^{\frac{1}{2}})}$$

$$\geq (NM)^{-1} e^{-5\,(N^{\frac{1}{2}}+M^{\frac{1}{2}})} S_N(\beta) S_M(\beta) \tag{6.15}$$

from which the existence of the limit Eq.(6.7) follows [9].

We finally add the following remarks concerning the length of $|\lambda|$ in the strip. If we define the "microcanonical" partition function corresponding to $S_N(\beta)$ by

$$Q_N(\varepsilon,\beta) = \sum_{(\lambda),\,|\lambda|=(1+\varepsilon)N}^{\sim} e^{-\mu(\lambda,\beta)} \tag{6.16}$$

then by an argument similar to that above one can show that

$$\sigma(\varepsilon,\beta) = \lim_{N\to\infty} N^{-1} \log Q_N(\varepsilon) \tag{6.17}$$

exists and is convex in ε and is related to $\tau(\beta)$ by the usual Legendre relation:

$$\tau(\beta 0 = -\beta + \sup_{0\leq\varepsilon}(-\beta\varepsilon + \sigma(\varepsilon,\beta) \tag{6.18}$$

If the sup is attained at a unique point $\varepsilon_0(\beta)$ then it is easy to see that $\frac{|\lambda|}{N} \to 1+\varepsilon_0(\beta)$ in probability as $N \to \infty$. It can be shown [6] that for small ε $\sigma(\varepsilon,\beta) = -\varepsilon\log\varepsilon + O(\varepsilon)$ and hence that $\varepsilon_0(\beta) = O(e^{-\beta})$, which gives a measure of how "straight" the line of separation is.

7 Concluding remarks

The technique of Section 2 can easily be applied to an antiferromagnetic Ising model with nearest neighbor interaction and zero external field to show that there is only one translationally invariant equilibrium state. It is known that there are at least two states which are not translationally invariant. In view of Dobrushin's result [4) that a small external field does not change the states drasti- cally one would expect that the uniqueness persists in the presence of such a field. As already mentioned the method of proof extends directly to 3 or more dimensions. It probably also extends to the case of an interaction between more than nearest neighbors if the neighbor interaction dominates the sum of the others. For a general ferromagnetic interaction the result is probably true, but it is not clear how to prove it by the technique used here.

Concerning the surface tension problem we remark that the generalization to 3 dimensions is not obviously straightforward. In this case it would be natural to consider *e.g.* the following boundary condition: Ω is a rectangular box of size $N \times N \times H$, and the upper half of it is surrounded by $+$ spins and the lower half by $-$ spins. Then a "big" surface of separation is present in every configuration, and its boundary is fixed. Its area will only exceed N^2 by a small amount $\delta(\beta)N^2$, but it is not easy to rule out that it "sticks to the boundary", so that a portion of order N^2 is located near it. That can cause trouble when one tries to carry through a subadditivity proof as that leading to Eq.(6.15) by joining together 4 boxes $N \times N \times H$ to one box $2N \times 2N \times H$.

The above problem is probably related to the problem of determining the magnitude of the fluctuations of the surface of separation with the following probability assumptions: Consider as above the class of surfaces having a fixed square boundary of size $N \times N$ and give to each surface λ the relative probability $e^{-\beta(\text{area of }\lambda)}$. One can then ask for the probability $P_q(n)$ that a vertical line through the point q in the square intersects the surface at height n. If one considers only the class of surfaces that intersect each vertical line only once one can prove that if β is large

$$P_q(n) \le (\sum_{k=4}^{\infty}(3e^{-\beta})^k k^2)^n \tag{7.1}$$

for all q, so the surface is very "rigid" in this case. The analogous problem in 2 dimensions, where λ becomes a line of separation, was considered by Temperley, [13]. In this case, it is easy to see that the successive vertical steps of λ are independent and equally distributed as random variables. The fluctuations far from the ends are therefore easily determined by the central theorem, and one sees that $P_q(n) \sim N^{\frac{1}{2}}$ if $n = O(N^{\frac{1}{2}})$ and the distance of q from the ends is $O(N)$. The "sur- face" has thus large fluctuations and is not "rigid" in 2 dimensions. As Temperley pointed out, if one computes the "surface tension" defined by:

$$\tau^{\sim} = \lim_{N\to\infty} N^{-1} \log \sum_{\lambda} e^{-\beta|\lambda|} = -\beta - \log\tanh\frac{\beta}{2} \tag{7.2}$$

one gets the value computed by Onsager for the Ising model, which is equal to τ, (see below). There is thus a cancellation of the errors involved in replacing the true class of paths by the restricted class and in neglecting the weight $\mu(\lambda, \beta)$ defined in Eq.(6.6). This cancellation is not really understood, except that one can check explicitly that the first few terms in an expansion in $e^{-\beta}$ are identical.

As mentioned above one can prove that our τ has the same value as that computed by Onsager from a grand canonical definition [14, 15]. Moreover, it is also equal to $\lim N^{-1} \log \frac{Z(M^{+-}(\Omega),\beta)}{Z(M^{++}(\Omega),\beta)}$ if $\delta > 1$ as can be seen by an explicit calculation, [15]. *A priori* this is not obvious because the big contour present in $M^{+-}(\Omega)$ is not prevented from being near the ends of the cylinder as it is in $M^{+-}(\Omega, m)$, and this could cause extra boundary effects. The fact that the two surface tensions are equal seems to indicate that the ratio of the probability of finding λ near the ends to that of finding it near the middle is not larger than $e^{o(N)}$.

The question of how much the phase boundary fluctuates is closely related to the question of the existence of non-translationally invariant equilibrium states. Consider the boundary condition just described. If λ fluctuates much a finite group of spins, $\{\sigma_{x_1}, \ldots, \sigma_{x_n}\}$ situated far from the boundary of Ω will be far from λ too with high probability and thus $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_\lambda$, given λ, will be equal to either $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_+$ or $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_-$, so the state will be a weighted average of these two states. If λ is very rigid, however, it can have a positive probability of going between some spins in the group, and $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_\lambda$ can take several other values with positive probability. Then $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle$ is a weighted average not only of the above two states. The previous discussion of the fluctuations supports the belief that in 2 dimensions there are only two extremal Gibbs states, whereas in 3 dimensions at low temperatures this is not the case.

1 Appendix to Section 2

Proof of Lemma 2.1. The second Griffiths inequality implies that the correlation $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+,\Omega}$ decreases when Ω increases [8]. Call Q_D a square centered at the barycenter of $x_1, \ldots, x_n$ having side $\frac{1}{2}$. Then

$$\begin{aligned} 0 \le & \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+,\Omega} - \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_+ \\ \le & \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+,Q_D} - \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_+ = f(x_1 \ldots x_n, D) \end{aligned}$$

The function $f(x_1 \ldots x_n, D)$ is translationally invariant (see Lemma 2.4) and decreases to zero as $D \to \infty$. A similar argument holds for $\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{-,\Omega}$.

Lemma 2.4: *The states $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\pm}$ are translationally invariant and verify the relations in Eq.(2.6),(2.7): Hence they are extremal translationally invariant and describe "pure phases". (See [2]).*

Proof. Let Q_D be defined as above and $Q_D + a$ be its translate by a. Obviously $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\mu, Q_D} = \langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{\pm, Q_D+a}$. Introduce also the square Q'_D having side D and a region Ω containing Q_D and $Q_D + a$ and lying inside of both Q'_D and $Q'_D + a$. (This is always possible if D is large enough.) Then as in the proof of Lemma 2.1 we have

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, Q'_D} \le \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, \Omega} \le \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, Q_D}$$

and

$$\begin{aligned} &\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, Q'_D} = \langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{+, Q'_D+a} \\ &\le \langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{+, \Omega} \le \langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{+, Q_D+a} = \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, Q_D} \end{aligned}$$

Hence

$$\begin{aligned} &|\langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{+, \Omega} - \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+}| \\ &\le \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, Q_D} - \langle \sigma_{x_1+a} \dots \sigma_{x_n+a} \rangle_{+, Q'_D+a} \le 2\, f(x_1 \dots x_n, D) \end{aligned}$$

and the transitional invariance of $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+}$ follows by passing to the limit $\to \infty$ in the last inequality. The proof of Eq.(2.6) is very easy. It is sufficient to remark that $\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+, \Omega} = (-1)^n \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{-, \Omega}$ because of the symmetry of the model under the transformation ax $\sigma_x \to -\sigma_x$. In order to prove Eq.(2.7) let us first recall that Griffith's inequality tells us that

$$\langle \sigma_{x_1} \dots \sigma_{x_n} \sigma_{y_1} \dots \sigma_{y_m} \rangle_{+} \ge \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+} \langle \sigma_{y_1} \dots \sigma_{y_m} \rangle_{+} \tag{A.1}$$

Suppose that Ω is a square box containing $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ and (assuming that the horizontal component of a tends to infinity) draw a vertical line vv separating the points $\{x_1, \dots, x_n\}$ from $y_1, \dots, y_m\}$. Let us fix the spins at the points of v to be $+1$. This corresponds to putting an infinite positive external field at the points of v. Then by Griffith's inequality the new correlation functions $\langle \sigma_{x_1} \dots \sigma_{y_m} \rangle^*_{\Omega}$ verify:

$$\langle \sigma_{x_1} \dots \sigma_{y_m} \rangle_{+, \Omega} \le \langle \sigma_{x_1} \dots \sigma_{y_m} \rangle^*_{\Omega} \tag{A.2}$$

But because the two regions Ω_1 and Ω_2 into which the box Ω is divided by v are independent we have:

$$\langle \sigma_{x_1} \dots \sigma_{y_m} \rangle^*_\Omega = \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{+,\Omega_1} \langle \sigma_{y_1} \dots \sigma_{y_m} \rangle_{+,\Omega_2} \tag{A.3}$$

From Eq(A.2) and (A.3) we obtain in the limit $\Omega \to \infty$:

$$\langle \sigma_{x_1} \dots \sigma_{y_m} \rangle_+ \le \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle^1_+ \langle \sigma_{y_1} \dots \sigma_{y_m} \rangle^2_+ \tag{A.4}$$

where the indices 1 and 2 indicate the presence of the boundary v. But when $d \to \infty$ we have:

$$\begin{aligned} &\langle \sigma_{x_1} \dots \sigma_{x_n} \rangle^1_+ \xrightarrow[d\to\infty]{} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_+ , \\ &\langle \sigma_{y_1} \dots \sigma_{y_m} \rangle^2_+ \xrightarrow[d\to\infty]{} \langle \sigma_{y_1} \dots \sigma_{y_n} \rangle_+ \end{aligned} \tag{A.5}$$

and as a consequence of Eq.(A.1), (A.4) and (A.5) we obtain (2.7).

Proof of Lemma 2.3. We have

$$\begin{aligned} P(\boldsymbol{\tau}, L^{\frac{4}{3}}) &= \sum_{\substack{\eta_1 \cdots \eta_s \\ \sum \eta_i | \ge L^{\frac{4}{3}}}} \sum_{\boldsymbol{\sigma}}{}^* \frac{w_{\boldsymbol{\tau}}(\boldsymbol{\sigma})}{Z^{\boldsymbol{\tau}}(\Omega, \beta)} \\ &= \sum_{\substack{\eta_1 \cdots \eta_s \\ \sum \eta_i | \ge L^{\frac{4}{3}}}} e^{-\beta \sum_i |\eta_i|} \frac{Z(M^+(\theta_1, \beta)) \dots Z(M^+(\theta_{s+1}, \beta))}{Z(M^+(\theta, \beta))} \end{aligned} \tag{A.6}$$

where we use the notations of Section 1 and observe that $Z(M^+(\tau), \beta) = Z(M^-(\theta), \beta)$. We shall establish the following inequalities

$$\sum_{\substack{\eta_1 \cdots \eta_s \\ \sum \eta_i | \ge L^{\frac{4}{3}}}} e^{-b \sum_i |\eta_i|} \le \left(\tfrac{4}{1-3e^{-\beta}} \right)^{2L} \max_{s \le 2L} \tbinom{L^{\frac{4}{3}}}{s} (3e^{-\beta})^{L^{4/3}} \tag{A.7}$$

$$Z(M^+(\theta_1), \beta) \dots Z(M^+(\theta_{n+1}), \beta) \le Z(M^+(\Omega), \beta) \tag{A.8}$$

$$Z(M^+(\Omega), \beta) \le e^{8L \frac{(3e^{-\beta})^4}{1-3e^{-\beta}}} Z(M^+(\Omega_1), \beta) \tag{A.9}$$

$$Z(M^{\boldsymbol{\tau}}(\Omega), \beta) \le e^{-6L} Z(M^+(\Omega_1), \beta), \tag{A.10}$$

where Ω_1 is a square box concentric with Ω having side $L - 2$. From the above inequalities we obtain Eq.(2.15) directly from Eq.(A.6) if $\beta > \log 3$. To prove Eq.(A.7) observe that the number of contours starting at a given point and having length ℓ is not larger than 3^ℓ, and furthermore we observe that if $\boldsymbol{\tau}$ is fixed also the number s of contours $\eta_1, \ldots, \eta_s$ is assigned, and there are at most $\binom{2s}{s}$ ways of choosing s starting points among the 2^s possible ones. Hence, since $2s \leq 4L$:

$$\sum_{\substack{\eta_1 \cdots \eta_s \\ \sum \eta_i | \geq L^{\frac{4}{3}}}} e^{-\beta \sum_i |\eta_i|} \leq \binom{2s}{s} \sum_{\substack{\ell_1, \ldots, \ell_s \\ \sum \ell_i \geq L^{4/3}}} (3e^{-\beta})^{\ell_1} \ldots (3e^{-\beta})^{\ell_s}$$
$$\binom{2s\cdot}{s}\binom{L^{4/3}}{s}(3e^{-\beta})^{L^{4/3}} \sum_{\ell'_1, \ldots \ell'_s \geq 0} (3e^{-\beta})^{\ell'_1} \ldots (3e^{-\beta})^{\ell'_1}$$
$$\leq (\frac{4}{1-3e^{-\beta}})^{2L} \max_{s \leq 2L} \leq \binom{2s}{s}(3e^{-\beta})^{L^{4/3}}$$

The inequality Eq.(A.8) is obvious from the definition of $Z(M^+(\Omega), \beta)$. The inequality Eq.(A.9) can be proved in the following way

$$\begin{aligned} Z(M^+(\Omega), \beta) &= \sum_{\gamma_1, \ldots, \gamma_n \subset \Omega} e^{-\beta \sum_i \gamma_i} \\ &\leq (\sum_{\gamma_1, \ldots, \gamma_n \subset \Omega_1} e^{-\beta \sum_i \gamma_i})(\sum_{\gamma'_1, \ldots, \gamma'_n \not\subset \Omega_1} e^{-\beta \sum_i \gamma'_i}) \\ &= Z(M^+(\Omega_1), \beta)(\sum_{\gamma'_1, \ldots, \gamma'_n \not\subset \Omega_1} e^{-\beta \sum_i \gamma'_i}) \end{aligned}$$

where $\gamma'_1, \ldots, \gamma'_n \not\subset \Omega_1$ means that none of the contours lies in Ω_1. This last sum is bounded by:

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{1}{r!}(\sum_{\gamma' \not\subset \Omega_1} e^{-\beta \sum_i \gamma'_i})^r &\leq \sum_{r=1}^{\infty} \frac{1}{r!}(8L \sum_{\gamma' \ni O} e^{-\beta |\gamma'_i|})^r \\ &\leq e^{8L \sum_{\ell=4}^{\infty} (3e^{-\beta})^{\ell}} = \exp 8L \frac{(3e^{-\beta})^4}{1-3e^{-\beta}} \end{aligned}$$

where $\sum_{\gamma' \ni O}$ denotes the sum over all contours containing a fixed point, and Eq.(A.9) follows. Finally, to prove Eq.(A.10) restrict the sum defining $Z(M^{\boldsymbol{\tau}}(\Omega), \beta)$ to a few terms, namely to those in which the contours $\eta_1, \ldots, \eta_s$ are fixed, while the contours $\gamma_1, \ldots, \gamma_n$ are put in Ω_1. We take the set of open contours to be such that they isolate the + spins of $\boldsymbol{\tau}$ and run parallel to the boundary of Ω and next to it. Then $\sum_i |\eta_i| \leq 6L$, and therefore Eq.(A.10) follows.

2 Appendix to Setion 5

Proof of Lemma 5.4. The proof uses the fact that the total number of − spins, $N(X)$, of a configuration X is the sum of contributions from the regions inside the outer contours (i.e. those not surrounded by any other contour). For a given configuration of outer contours, Γ, these contributions are independent random variables and those coming from congruent regions have equal distributions. Thus $N(X)$ fluctuates for two reasons, one because the number of outer contours in each congruence class (γ), $K_{(\gamma)}(\Gamma(X))$, is random, and the other because the contributions from the regions belonging to the various congruence classes are also random. The fluctuations for given outer contours can easily be estimated using the independence of the contributions. The fluctuations in the $K_{(\gamma)}$ are more complicated to estimate, and we consider them first. We want to estimate the deviations $S_{(\gamma)}(\Gamma(X))$ defined by:

$$|\theta|^p S_{(\gamma)}(\Gamma(X)) = K_{(\gamma)}(\Gamma(X)) = \langle K_{(\gamma)}(\Gamma(X))\rangle_{M^+_{0,c}(\theta)} \tag{B.1}$$

The contribution to $N(X)$ due to the fact that $S_{(\gamma)} \neq 0$ is clearly bounded by:

$$|\theta|^p \sum_{(\gamma)} (\text{area of } \gamma)|S_{(\gamma)}| \leq |\theta|^p \sum_{(\gamma)} |\gamma|^2 S_{(\gamma)}| \equiv |\theta|^p s \tag{B.2}$$

and therefore we want to estimate the fluctuations of $S(\Gamma(X))$ defined in (B.2). This can be done using the following "saddle point" technique, which is often useful for the probability distributions of "exponential type" common in statistical mechanics. (We suggest, however, that the reader first studies how the theorem follows from the estimate after Eq.(B.23)).

The probability in $M^+_c(\theta)$ of a possible configuration of outer contours, G, is given by:

$$\begin{aligned} P_\mu(\Gamma) &= \prod_{\gamma\in\Gamma} e^{\mu(\gamma)} \frac{Z^-_c(\gamma,\mu)}{Z^+_{0,c}(\theta,\mu)} \\ &= \prod_{(\gamma)} \frac{\left(e^{\mu(\gamma)} Z^-_c(\gamma,\mu)\right)^{K_{(\gamma)}(\Gamma)}}{Z^+_{0,c}(\theta,\mu)} \end{aligned} \tag{B.5}$$

where $Z^-_c(\gamma,\mu)$ denotes the partition function of the ensemble of configurations in the region enclosed by γ such that all spins along the inside of γ

are -1. We consider for the moment an arbitrary translationally invariant weight $\mu(\gamma)$ instead of $-\beta|\gamma|$. $P_\mu(\gamma)$ is thus of "exponential type"

$$P_\mu(\Gamma) = e^{\sum_{(\gamma)} a_{(\gamma)}(\mu) K_{(\gamma)}(\Gamma) - f(a(\mu))} \tag{B.6}$$

with

$$a_{(\gamma)}(\mu) = \mu(\gamma) + \log Z_c^-(\gamma, \mu) \tag{B.7}$$

and

$$e^{f(a(\mu))} = \sum_\Gamma e^{\sum_{(\gamma)} a_{(\gamma)}(\mu) K_{(\gamma)}(\Gamma)} = Z(M_{0,c}^+(\theta), \mu). \tag{B.8}$$

The generating function $\langle e^{\sum_{(\gamma)} \Delta a_{(\gamma)} K_{(\gamma)}} \rangle_\mu$ can thus be written for any numbers $\Delta a_{(\gamma)}$:

$$\langle e^{\sum_{(\gamma)} \Delta a_{(\gamma)} K_{(\gamma)}} \rangle_\mu = e^{f(a+\Delta a) - f(a)} \tag{B.9}$$

For any possible values $K_{(\gamma)}$ we thus have the following inequality:

$$P_\mu(K_{(\gamma)}) e^{\sum_{(\gamma)} \Delta a_{(\gamma)} K_{(\gamma)}} \leq e^{f(a+\Delta a) - f(a)} \tag{B.10}$$

for all Δa. By a judicious choice of Δa we can thus estimate $P_\mu(K_{(\gamma)}$ by:

$$P_\mu(K_{(\gamma)}) \leq e^{f(a+\Delta a) - f(a)}\, e^{-\sum_{(\gamma)} \Delta a_{(\gamma)} K_{(\gamma)}} \tag{B.11}$$

For $K_{(\gamma)}$ near the average $\langle K_{(\gamma)} \rangle_\mu$ we can find a good choice of $\Delta a_{(\gamma)}$ by considering the first two terms in the Taylor expansionof th exponent in Eq.(B.11). We have $K_{(\gamma)} = \frac{\partial f(a(\mu))}{\partial a(\gamma)} + |\theta|^p S_{(\gamma)}$ for $\mu(\gamma) = \beta|\gamma|$, and put $\Delta a(\gamma) = a_{(\gamma)}(\mu + \Delta\mu) - a_{(\gamma)}(\mu)$ and expand to second order in $\Delta\mu(\gamma))$. The 0-th order term vanishes, the first order is $-\sum_{(\gamma)} |\theta|^p S_{(\gamma)} da_{(\gamma)}(\mu)$, and the second order term is $\frac{1}{2}(d^2 f(a(\widetilde{\mu})) - \sum_{(\gamma)} K_{(\gamma)} d^2 a_{(\gamma)}(\widetilde{\mu}))$ for some $\widetilde{\mu}$ between μ and $\mu + \Delta\mu$ ($da, d^2 a$ are the variation of $a(\mu)$ and its square corresponding to the variation of μ). Since

Consider the second order term first

$$f(a(\mu)) = \log Z(M_{0,c}^+(\theta), \mu) = \log \sum\nolimits_{\gamma_1, \ldots, \gamma_n \subset \theta}^{c} e^{\sum_1^n \mu(\gamma_i)} \tag{B.12}$$

we see, as explained in Eq.(4.40) that

$$d^2f(a(\widetilde{\mu})) = \sum\nolimits_{\gamma\subset\theta}^{c} (\Delta\mu(\gamma))^2[\widetilde{\rho}_{\theta.c}(\gamma) - \widetilde{\rho}^2_{\theta,c}(\gamma)] + \sum_{\substack{\gamma_1,\gamma_2\in\theta \\ \gamma_1\neq\gamma_2}}^{c} \Delta\mu(\gamma_1)[\widetilde{\rho}_{\theta,c}(\gamma_1,\gamma_2) - \widetilde{\rho}_{0,c}(\gamma_1)\widetilde{\rho}_{\theta,c}(\gamma_2)]. \tag{B.13}$$

We now make the "judicious choice":

$$\Delta\mu(\gamma) = \begin{cases} \operatorname{sign}(S_{(\gamma)})\,|\gamma|^2\,t & \text{if } |\gamma| \le c\log|\Omega| \\ 0 & \text{otherwise} \end{cases} \tag{B.14}$$

for some t with $0 \le t \le (c\log|\Omega|)^{-1}$.

Tnis restriction ensures that $|\Delta\mu(\gamma)| \le |\gamma|$, so that $\mu(\gamma) + \Delta\mu(\gamma) \le -(\beta-1)|\gamma|$ and $\widetilde{\mu}(\gamma) \le -(\beta-1)|\gamma|$. Then we can estimate Eq(B.13) using Eq.(4.61) with $b = (\beta-1)$ and get:

$$\frac{1}{2}d^2f \;\le\; (2\cdot 10^{12})e^{-(\beta-1)t^2|\theta|\sum_{(\gamma)}|\gamma|^2e^{-\frac{1}{2}(\beta-1)|\gamma|}} \tag{B.15}$$

if $3e^{-(\beta-\frac{1}{2})} \le \frac{1}{2}$. The last sum can be estimated using Lemma 4.1: $\sum_{(\gamma)}\cdot \le 16$, so we get:

$$\frac{1}{2}d^2f \le 10^{14}e^{-\beta}|\theta|t^2 \tag{B.16}$$

if $3e^{-(\beta-\frac{1}{2})} \le \frac{1}{2}$. Similarly $d^2a(\gamma)(\widetilde{\mu})$ can be expressed as a quadratic form similar to Eq.(B.13) with the correlation functions in the ensemble inside γ mentioned above. $d^2a(\gamma)(\mu)$ is thus non-negative, because the quadratic form is non-negative definite, and $\sum_{(\gamma)}K_{(\gamma)}d^2a_{(\gamma)}(\widetilde{\mu}) \ge 0$

Consider now the first order term. In it we have

$$da_{(\gamma)}(\mu) = \Delta\mu(\gamma) + d\,\log Z_c^-(\gamma,\mu) = \Delta\mu(\gamma) + \sum_{\gamma_1\ inside\ \gamma}^{c} \Delta\mu(\gamma_1)\rho^-_{\gamma,c}(\gamma_1) \tag{B.17}$$

where $\rho^-_{\gamma,c}(\gamma_1)$ is the correlation function in the ensemble inside γ mentioned above. As shown in Eq.(4.41) we have $\rho^-_{\gamma,c}(\gamma_1) \le e^{-\beta|\gamma_1|}$, so the sum can be estimated by:

$$\sum_{\gamma_1\ inside\ \gamma} t|\gamma_1|^2e^{-\beta|\gamma_1|} \le \frac{t|\gamma|^2}{16}\sum_{\ell=4}^{\infty}\ell^2(3e^{-\beta})^\ell \le t|\gamma|^2e^{-2\beta} \tag{B.18}$$

if $3e^{-\frac{1}{2}\beta} \leq \frac{1}{2}$. Its contribution is thus estimated by:

$$\sum_{(\gamma)} |\theta|^p |S_{(\gamma)}|\, t\, |\gamma|^2 e^{-2\beta} = t|\theta|^p e^{-2\beta} S \leq \frac{t\,|\theta|^p\, S}{2} \quad \text{if } 3e^{-\frac{\beta}{2}} \leq \frac{1}{2},$$

and finally

$$-\sum_{(\gamma)} |\theta|^p S_{(\gamma)} d\, a_{(\gamma)}(\mu) \leq -\sum_{(\gamma)} |\theta|^p |S_{(\gamma)}|\, |\gamma|^2 + \frac{t|\theta|^p}{2} \leq -\frac{2|\theta|^p S}{2} \tag{B.19}$$

Forgetting the negative second order term and choosing t in an optimal way we thus get the following bound for $P(K_{(\gamma)})$:

$$P(E_{(\gamma)}) \leq \exp \min_{0 \leq t \leq (c\log|\Omega|)^{-1}} [-\frac{t|\theta|^p S}{2} + \delta(\beta)|\theta|t^2], \tag{B.20}$$

where we have put $\delta(\beta) = 10^{14} e^{-\beta}$.

The minimum occurs for $t = \frac{S\,|\theta|^p}{4|\theta|\delta(\beta)}$ if this quantity is $\leq (c\,\log|\Omega|)^{-1}$. So we finally get:

$$P(E_{(\gamma)}) \leq \exp -\frac{S^2 |\theta|^{2p-1}}{16\delta(\beta)} \tag{B.21}$$

for $\leq 4\delta(\beta)|\tau|^{1-p}(c\log|\Omega|)^1$ and $3e^{-(\beta-\frac{1}{2})} \leq \frac{1}{2}$. To estimate the fluctuations in S we first note that $K_{(\gamma)} \leq |\theta|$ if $P(K_{(\gamma)}) \neq 0$, so the number of sequences $\{K_{(\gamma)}\}$ with $P(K_{(\gamma)}) \neq 0$ is bounded by:

$$|\theta|^{\sum_{\ell=4}^{c\log|\Omega|} 3\ell} \leq \exp(2\log|\Omega|)|\Omega|^{c\log 3}.$$

We can thus conclude that

$$\begin{aligned} P(S(\Gamma(X)) \geq T) &\leq \sum_{\substack{\{K_{(\gamma)}\} \\ S(\Gamma)\geq T}} \exp\min[\ldots] \\ &\exp((2\log|\Omega|)|\Omega|^{c\log 3} - \frac{T^2|\theta|^{2p-1}}{16\delta(\beta)}) \end{aligned} \tag{B.22}$$

if $T \leq 4\delta(\beta)|\theta|^{1-p}(c\log|\Omega|)^{-1}$, because $\min[\ldots]$ is a decreasing function of S as is easily checked. If $|\theta| > |\Omega|$ and $2p - 1 > c\log 3$, which is true if $p > \frac{1}{2}(1 + c\log 3)$, and if $T^2/\Delta(\beta)$ is bounded below, *e.g.* by 1 then the negative term dominates when $|\theta|$ is large, so we can finally conclude that the following estimate is valid:

$$P_{M^+_{0,c}(\theta)}(S(\Gamma(X)) \geq T) \leq \exp -\frac{T^2|\theta|^{2p-1}}{20\delta(\beta)} \tag{B.23}$$

if $\delta(\beta)^{\frac{1}{2}} \leq T \leq 4\delta(\beta)|\theta|^{1-p}(c\log|\Omega|)^{-1}$ for some $k > 0$, $3^{-(\beta-\frac{1}{2})} \leq \frac{1}{2}$ and $|\theta|$ large.

We now come to the study of the fluctuations in $N(X)$ for a given configu- $(\gamma_1, \ldots, \gamma_k)$ of outer contours with "occupation numbers" $\{K_{(\gamma)}$. We make use of the following straightforward estimate:

Lemma 5A.1:

Let $n_1, \ldots, n_k$ be independent random variables which are all bounded, $|n_i| \leq B_i$, $i = 1, \ldots, k$ and let $N = \sum_i n_i$, $B^2 = \sum_{i=1}^k B_i^2$. Then

$$P(|N - \langle N\rangle| \geq a) \leq 2e^{-\frac{a^2}{2B^2}}. \tag{B.24}$$

Proof. Let $f_i(t) = \log(e^{tn_i})$ and $f(t) = \log\langle e^{tN}\rangle = \sum_{i=1}^k f_i(t)$, and denote "canonical averages" by $\langle g(n_i)\rangle_t \overset{def}{=} \frac{\langle g(n_i)e^{tn_i}\rangle}{\langle e^{tn_i}\rangle}$. Since $\langle N\rangle = f'(0)$ we have $\langle e^{t(N-\langle N\rangle)}\rangle = e^{f(t)-tf'(0)}$, and as in the previous argument we get the inequality $e^{ta}P((N - \langle N\rangle) > a) \leq e^{f(t)-tf'(0)}$ for any $t \leq 0$, so that for a judicious choice of t we get:

$$P(|N - \langle N\rangle| \geq a) \leq e^{f(t)-tf'(0)-ta}. \tag{B.25}$$

If we expand the exponent to second order in t we get $-ta + \frac{1}{2}t^2 f^{\sim}(\widetilde{t})$ for some $\widetilde{t}$ between 0 and t. The $f^{\sim}(\widetilde{t})$ can be written as $\sum_{i=1}^k \langle n_1^2\rangle_{\widetilde{t}} - \langle n_i\rangle_{\widetilde{t}}^2$ and is thus bounded by $\sum_{i=1}^k B_i^2 = B^2$ for any $\widetilde{t}$. We thuas see that:

$$P(|N - \langle N\rangle| \geq a) \leq \exp\min_{t\geq 0}(\frac{1}{2}B^2t^2 - at) = e^{-\frac{a^2}{2B^2}}, \tag{B.26}$$

and the "judicious choice" is $t = a/B^2$.

In the same way we see that

$$P(|N - \langle N\rangle|) \leq e^{-\frac{a^2}{2B^2}}, \tag{B.27}$$

and the lemma is proved.

In our context n_i is the contribution to $N(X)$ from the region inside γ_i. It is clearly bounded by the area of γ_i, which is bounded by $B_i = \frac{|\gamma_i|^2}{16}$, and we get $B^2 = \sum_i \frac{|\gamma_i|^4}{16^2} = \sum_{(\gamma)} K_{(\gamma)} \frac{|\gamma|^4}{16^2}$. We thus see (since $|\gamma| \le c \log|\Omega|$ if $K_{(\gamma)} \neq 0$) that

$$\begin{aligned} B^2 &\le \sum_{(\gamma)} (K_{(\gamma)}) \frac{|\gamma|^4}{16^2} + \frac{(c\log|\Omega|)^2}{16^2} |\theta|^p \sum_{(\gamma)} |\gamma|^2 |S_{(\gamma)}| \\ &= \sum_{\gamma\subset\theta} \pi_{\theta,c}(\gamma) \frac{|\gamma|^4}{16} + \frac{(c\log|\Omega|)^2 |\theta| S}{16^2} \\ &\le \frac{|\theta|}{16^2} [\sum_{\ell=4}^{\infty} \ell^4 (3e^{-\beta})^\ell + (c\log|\Omega|)^2 |\theta|^{p-1} S] \end{aligned} \tag{B.28}$$

using the estimate $\pi_{\theta,c}(\gamma) \le e^{-\beta|\gamma|}$, Eq.(4.41), of the probability that γ is an outer contour. This means that if we consider configurations Γ satisfying e.g. $S(\Gamma) \le \delta(\beta)|\theta|^{1-p'}$ for some p' with $p < p' < 1$ then the last term becomes uniformly small for $|\theta| > k|\Omega|$ large, so we can say that $B^2(\Gamma) \le \delta(\beta)|\theta|^{1-p'}$, $3e^{-\frac{1}{2}\beta} \le \frac{1}{2}$, $|\theta| > k|\Omega|$ and $|\theta|$ is large. For any such configuration we thus conclude from the lemma that

$$P(|N(X) - \langle N(X)\rangle_p| > \frac{t|\theta|^p}{2}) \le 2e^{-\frac{t^2|\theta|^{2p}}{8\,\delta(\beta)|\theta|}} \tag{B.29}$$

If now we consider values of t such that $\delta(\beta)^{\frac{1}{2}}(\beta) \le \frac{1}{2}t \le \delta(\beta)|\theta|^{1-p'}$ and Γ such that $S(\Gamma) \le \frac{1}{2}t$ then, since

$$\begin{aligned} |\langle N(X)\rangle_\Gamma - \langle N(X)\rangle_{M^+_{0,c}}| \le& \sum_{(\gamma)} (\text{area of } \gamma) K_{(\gamma)}(\Gamma) - \langle K_{(\gamma)}(\Gamma(X))\rangle_{M^+_{0,c}}| \\ \le& |\theta|^p S(\Gamma) \le \frac{1}{2} t|\theta|^p, \end{aligned} \tag{B.30}$$

it follows that $|N(X) - \langle N(X)\rangle_{M^+_{0,c}(\theta)}| \ge t|\theta|^p$ implies $|N(X) - \langle N(X)\rangle_\Gamma| \ge \frac{1}{2}t|\theta|^p$, we see that

$$P(|N(X) - \langle N(X)\rangle_{M^+_{0,c}(\theta)}| \ge t|\theta|^p|\Gamma|) \le 2e^{-\frac{t^2|\theta|^{2p-1}}{8\delta(\beta)}} \tag{B.31}$$

uniformly in Γ and t. Summing over all possible Γ we then get the following estimate using Eq.(B.23):

$$
\begin{aligned}
&P_{M_{0,c}^{+}(\theta)}(|N(X)-\langle N(X)\rangle_{M_{0,c}^{+}(\theta)}|\geq t|\theta|^{p})\leq P_{M_{0,c}^{+}(\theta)}(S(\Gamma(X))\geq\frac{1}{2}t)\\
&+\sum_{\Gamma_i,S(\Gamma)\leq\frac{t}{2}}P_{M_{0,c}^{+}(\theta)}(\Gamma)2e^{-\frac{t^2|\theta|^{2p-1}}{8\delta(\beta)}}\leq 3e^{-\frac{t^2|\theta|^{2p-1}}{80\delta(\beta)}}
\end{aligned}
\tag{B.32}
$$

if, *e.g.*, $p'=\frac{1}{2}(1+p)$, $1>p>\frac{1}{2}(1+c\log 3)$, $\delta(\beta)^{\frac{1}{2}}\leq\frac{1}{2}t\leq\delta(\beta)|\theta|^{1-p'}$, $3e^{-(\beta-\frac{1}{2})}\leq\frac{1}{2}$, $|\theta|>k|\Omega|$ and $|\theta|$ large. (Note that the last restriction on t implies the one needed to apply Eq.(B.23)).

To prove Lemma 5.4 we finally estimate the difference $\langle N(X)\rangle_{M_{0,c}^{+}(\theta)}-|\theta|\frac{1-m^*}{2}$. Using the estimate $\rho(\gamma)\leq e^{-\beta|\gamma|}$, see Eq.(4.41), valid in the ensemble $M_0^{+}(\theta)$ we also get:

$$
\begin{aligned}
1-&P_{M_0^{+}(\theta)}(M_{0,c}^{+}(\theta))\leq\sum_{\substack{\gamma\subset\theta\\|\gamma|\geq c\log|\Omega|}}\leq|\theta|\sum_{\ell=c\log|\Omega|}(3e^{-\beta})^{\ell}\\
\leq&2|\theta|(3e^{-\beta})^{c\log\|\Omega|}\leq 2|\Omega|^{1+c\log(3e^{-\beta})}
\end{aligned}
\tag{B.33}
$$

if $3e^{-(\beta-\frac{1}{2})}\leq\frac{1}{2}$, so it goes to zero if β is large, and because

$$
\begin{aligned}
&\langle N(X)\rangle_{M_0^{+}(\theta)}=P_{M_0^{+}(\theta)}(M_{0,c}^{+}(\theta))\langle N(X)\rangle_{M_{0,c}^{+}(\theta)}\\
&+(1-P_{M_0^{+}(\theta)}(M_{0,c}^{+}(\theta)))\langle N(X)\rangle_{\overline{M}_{0,c}^{+}(\theta)}
\end{aligned}
\tag{B.34}
$$

we get

$$
\begin{aligned}
&|\langle N(X)\rangle_{M_0^{+}(\theta)}-\langle N(X)\rangle_{M_{0,c}^{+}}|\leq 4|\theta||\Omega|^{1+c\log(3e^{-\beta})}\\
&\leq 4|\Omega|^{2+c\log(3e^{-\beta})}\leq 4|\Omega|^{\frac{1}{2}}
\end{aligned}
\tag{B.35}
$$

e.g. if $3e^{-\beta}\leq e^{-\frac{3}{2}c}$, *i.e.* if β is large. Moreover, it is shown in [6] that

$$
|\langle N(X)\rangle_{M_0^{+}(\theta)}-|\theta|\frac{(1-m^*)}{2}|\leq\mathit{const}\,|\partial\theta|
\tag{B.36}
$$

if $3e^{-(\beta-\frac{1}{2})}\leq\frac{1}{2}$. Since $\frac{|\Omega|^{\frac{1}{2}}}{t|\theta|^{p}}\to 0$ as $|\theta|\to\infty$, for the values we consider these estimates show that for $|\theta|$ large

$$
P_{M_{0,c}^{+}(\theta)}(|N(X)-|\theta|\frac{1}{2}(1-m^*)|\geq t|\theta|^{p})\leq 3e^{-\frac{t^2|\theta|^{2p-1}}{100\delta(\beta)}}
\tag{B.37}
$$

when the above restrictions are fulfilled, and Lemma 5.4 is proved because the magnetization is $|\theta| - 2N(X)$.

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