Lettere al Nuovo Cimento Vol.**2**, Pag. 143-146 24 Luglio 1971

Surface tension in the Ising model

D. Abraham ([1]) G. Gallavotti([2])(*) A. Martin-Löf([3])

([1]) *Massachussets Institute of Technology - Cambridge, Mass.*
([2]) *Istituto di Fisica dell'Università - Padova*
([3]) *Rockefeller University - New York, N.Y.*

(ricevuto 7 Giugno 1971; approvato)

In this note we consider a 2–dimensional Ising ferromagnet with nearest neighbor coupling $J = -\frac{1}{2}$. We suppose the system enclosed in a box Ω periodicized in one direction (*i.e.* Ω is regard as a cylinder). $|\Omega|$ denotes its area.

Suppose that $\beta = 1/kT$ is large enough so that a spontaneous magnetization m^* is possible in zero external field.

Let us ask how it is possible to describe, from a microscopic point of view, the phenomenon of coexistence of two oppositely magnetized phases and how one can give a purely microscopic definition of surface tension ([1]). Minlos and Sinai have first developed a powerful technique to deal with this type of questions ([2]). In this paper we describe the results we obtained on the surface tension problem by using their methods nd ides we shall systematically skip the proofs which will be published elsewhere.

Suppose the spins on the upper base of the cylinder Ω are forced, by an external field, to be up and the spins on the lower base are forced to be down. Suppose we consider the model in th canonical ensemble in which the magnetization is fixed to be $m = \alpha m^* + (1-\alpha)(-m^*) = (2\alpha - 1)m^*$, $0 < \alpha < 1$. Finally suppose that, for a given spin configuration, we draw a line of length one at mid distance between every couple of nearest-neighbor opposite spins. A moment of reflection shows that, as a consequence of the assumed boundary condition, the set of lines thus obtained splits into several closed, disjoint, self-avoiding contours ([3]). Then, once can prove that, picking up of the canonical ensemble a random configuration of spins, the set of contours $\gamma_0, \gamma_1, \ldots, \gamma_n$ associated with this configuration will be, with a probability tending to one as $\Omega \to \infty$ if β is large enough, such that

i) $\gamma_1, \ldots, \gamma_n$ will have a length $|\gamma_i| \le c_0 \log |\Omega|$ for suitable $c_0 > 0$.

(*) Permanent address: Istituto di Matematica dell'Università, Roma.
([1]) The usual microscopic approach to the surface tension is reviewed in the excellent review article: S. Ono and S. Kondo: *Handbuch der Physik*, Vol. **10**, (Berlin, 1960), p.134.
([2]) R.A. Minlos and Y.G. Sinai: *Trans. Moscow. Math. Soc.,* **19***,121 (1968) (English translation); Mar. USSR Sbornik,* **2***, 335 (1967).*
([3]) Strictly speaking the contours@= can have some corners (but not ides) in common; this gives rise to some ambiguity in recognizing which, in fact, are the two contours meeting at a corner, One has to choose, once and for all, one of the possible prescriptions to resolve this ambiguity.

ii) γ_0 will be long and will go around the cylinder. If N is the length of the bases of Ω, then $|\gamma_0| \le (1 + O(e^{-\beta})N$ (*i.e.* γ_0 will be almost straight at low temperature).

iii) γ_0 will be located at an height αH, if H is the height of Ω. The volume of the region above γ_0 will be $\alpha|\Omega|$ with a fluctuation not exceeding $|\Omega|^{3/4}$ (hence very small).

iv) The average magnetization in the region above γ_0 will be m^* and the average magnetization of the region below γ_0 will be $-m^*$. The total magnetization in these two regions will be, respectively, $m^*\alpha|\Omega|$, $-m^*(1-\alpha)|\Omega|$ with a fluctuation not exceeding $|\Omega|^{3/4}$ (hence very small).

In other words we, typically, have two regions separated by a random line (quite well defined in shape and position). The region above γ_0 consists of up spins with a lot of small holes containing down spins (the linear dimensions of these holes are not larger than $c_0 \log|\Omega|$, which are just enough to make the average magnetization of this region $= m^*$: the region below γ_0 looks like the one above it, except that the roles of the up and down spins are interchanged.

It is very natural to interpret the line γ_0 the surface of separation of two coexisting phases with opposite magnetization.

Thin is confirmed by the fact that one can also prove that, forcing the spins on the bases to he all up, then a typical configuration in the canonical ensemble with magnetization m^* will simply consist of contours $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\gamma_i| \le c_0 \log|\Omega|$ (i.e. the situation describes a pure phase with positive magnetization).

Call $Z^{+-}(\Omega, \beta, m)$ the canonical partition function, at temperature β^{-1}, associated with the boundary condition in which the spins on the upper boundary are up and the ones on the lower boundary are down (here, as above, $m = (2\alpha - 1)m^*$, $0 < \alpha < 1$). Let $Z^{++}(\Omega, \beta, m^*)$ be the canonical partition function associated with a magnetization $m = m^*$ and with a boundary condition in which all the spins on the bases are up.

In view of the results described above, it is very natural to identify the quantity

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{+-}(\Omega.\beta, m)}{Z^{++}(\Omega.\beta, m^*)} \tag{1}$$

as the surface tension between the two phases.

We show ([4]) that the above limit exists and, in fact, it can be put in the form of a one-dimensional thermodynamic limit for a suitable system whose configurations can be interpreted as the set of possible shapes of the surface γ_0 of separation between the two phases. We also prove that, with a large probability, the length of the surface of separation is (for a suitable $\varepsilon(\beta)$) ([4])

$$|\gamma_0| = (1 + \varepsilon(\beta))N, \qquad \varepsilon(\beta) = O(e^{-\beta}) \tag{2}$$

here ≪large≫ means that the probability that $||\gamma_0 - (1 + \varepsilon(\beta))N| > \xi N$ is exponentially small $\forall \xi > 0$, while the probability that $||\gamma_0| - (1 + \varepsilon(\beta))N| < \xi N$ does not tend to zero faster than a power in N, for all $\xi > 0$. This last result is an improvement of typical results of Minlos and Sinai.

([4]) G. Gallavotti, A. Martin-Löf: *Surface tension in the Ising Model*, Comm. Math. Phys., **25**, 87–126, 1972.

Although we have confined our attention to the 2-dimensional Ising model, the technique is dimension independent. This is no longer true for the results described below.

Considering the 2-dimensional nearest-neighbor Ising model we have shown ([5]) that the value (1) can be exactly computed and and coincides Onsager's value ([6])

$$\tau(\beta) = -\beta + 2\,\mathrm{arctgh}\, e^{-\beta} \tag{3}$$

In ref. ([5]) we have also shown that $\tau(\beta)$ given by (3) and (1) can be defined through the grand canonical ensemble partition function as

$$\begin{cases} Z^{+-}(\Omega,\beta) = \sum_m Z^{+-}(\Omega,\beta,m), \\ Z^{++}(\Omega,\beta) = \sum_m Z^{++}(\Omega,\beta,m), \end{cases} \tag{5}$$

This last result is probably true in general, but we have been able to prove it only by explicit calculation in the 2-dimensional model.

In ref.([4]) the proof of the phase separation phenomenon described at p. 144 is obtained by adapting to our situation (i.e. to our boundary conditions) the work of Minlos and Sinai and should be quite obvious to the reader familiar with their work. Once proved the results about the description of a typical configuration, we found a convenient expression for the ratio of the partition functions appearing in (1) by combining the Minlos and Sinai technique on the contour correlation functions Bellmans' technique to compute the surface effect due to a fixed boundary; this reduced the proof of the existence of the limit (1) to a certain 1-dimensional. thermodynamic limit problem which we solved.

The technical conditions under which we have been able to prove the above results, are:

1) β must be large (in fact much larger than β_c),

2) the height H of Ω is such that $N = N^\delta$, $\delta > 1$, id N is the length of the base of Ω.

Condition 2) has a clear physical meaning; since the width of γ_0, though very small compared to N, can be of the order of N (see (2)), therefore one has to let H grow faster than N in order to avoid that the phase separation surface comes too close to the base of Ω for small α.

In ref. ([5]) we prove the equality of (1) and (3) by using the method of ref. ([2]) combined with Dobrushin's technique ([7]) to prove the existence of a phase transition in a Ising antiferromagnet. This result is again obtained under the same technical conditions 2) and 1) above. In ref. ([5]) is also proven the equality of (3) and (4); this is accomplished by explicitly computing (4) using the standard techniques ([8]) for the exact solution of the Ising model together with the assumption 2) above (only).

we stress the difference between the surface tension and the interface and boundary effects studied by other authors ([9]) ([10]); the quoted authors were interested in computing

([5]) D. Abraham, G. Gallavotti and A. Martin-Löf: *Surface tension in the two–dimensional Ising model*, Physica, **65**, 73–88, 1973.

([6]) L. Onsager, *Phys. Rev.*, **65**, 117, 1944.

([7]) R.L. Dobrushin: *Funct. Anal. Appl.*, **2**, 302, (1967) (English translation).

([8]) T.D. Schultz, D.C. Mattis and E.H. Lieb, *Rev. Mod. Phys.*, **36**, 956, (1964).

([9]) M.E. Fisher and A.E. Ferdinand, *Phys. Rev. Lett.*, **19**, 169 (1967).

([10]) A. Bellemans: *Physica*, **28**, 493, 617 (1962); **29**, 548 (1963).

the surface tension between the system and a *fixed* boundary. Here we are faced with the problem of evaluating the surface tension phases and then to find the effect of a *random* boundary,

From our theory there are strong indications that in general $\tau(\beta) - \beta$ is analytic in $e^{-\beta}$ and it would be interesting to find a Meyer expansion technique for the series of $\tau(\beta) - \beta$ in powers of $e^{-\beta}$; in fact, from the point of view of the equivalent 1–dimensional thermodynamic limit problem, the variable $e^{-\beta}$ plays the role of the activity.

The details of the proofs needed to obtain the above results will be published in two forthcoming papers already available in form of preprints ([4,5]).

* * *

The authors are indebted to O. Lanford, J.L. Lebowitz, E.H. Leb and S. Miracle-Solé for enlightening discussions.