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Dynamics of a Local Perturbation in the *XY* Model. I—Approach to Equilibrium

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The magnetisation and spin correlation functions of the X–Y model are analysed exactly when the system is subjected to a local perturbation of the magnetic field in the z-direction. When such a perturbation is removed, these quantities approach their equilibrium values asymptotically as $(\text{time})^{-1}$.

1. Introduction

The X–Y model [1] is an example of a quantum-mechanical many-body system of spins interacting in one dimension. Provided the interactions are restricted to nearest neighbours, many equilibrium properties for the canonical ensemble may be obtained exactly [1, 2]. Recently, quantities have been investigated whose time dependence arises in the following way: initially the system is described by an equilibrium canonical density. The parameters of the system such as z-direction magnetic field are subsequently allowed to vary in time, the system thereby evolving according to the quantum-mechanical equations of motion. It is pertinent to inquire whether the system approaches a new equilibrium state under such perturbations. It is known that for a homogeneous time-dependent field, the magnetization in the same direction and the spin correlation functions approach time-independent values which do not correspond to equilibrium at any temperature [3]. Tjon [4] has studied the case when a time-dependent field is applied to a boundary spin in the weak-coupling approximation. He found approach to equilibrium as $(\text{time})^{-3}$. Girardeau [5] has performed a numerical study which showed approach to equilibrium of the magnetization.

In this paper we obtain the exact time-dependent magnetization for any spin inside a chain subjected to a local interior perturbation. We find that any spin (in the isotropic case as well as the anisotropic case, with the exception of the degenerate Ising limit) approaches equilibrium asymptotically as $(\text{time})^{-1}$.

Furthermore, for the isotropic case we show that all the correlation functions approach their correct equilibrium values as $(\text{time})^{-1}$.

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2. Formulation

The Hamiltonian which describes the system under consideration is given by

$$\mathscr{H} = \mathscr{H}_0 + h(t)\sigma_m^z \tag{2.1}$$

where $\mathscr{H}_0$ is the usual X–Y Hamiltonian

$$\mathscr{H}_0 = \frac{1}{4}\sum_1^M ((1+\gamma)\sigma_m^x\sigma_{m+1}^x + (1-\gamma)\sigma_m^y\sigma_{m+1}^y). \tag{2.2}$$

The σ^α are Pauli spin operators, $h(t)$ is the external magnetic field and γ is the measure of anisotropy. We choose cyclic boundary conditions so that

$$\sigma_{M+1}^\alpha = \sigma_1^\alpha. \tag{2.3}$$

For the relaxation problem, we have

$$h(t) = \begin{cases} h \geq 0, & t \leq 0 \\ 0 & t > 0 \end{cases} \tag{2.4}$$

and the initial density operator is $\rho = Z^{-1}\, e^{-\beta\mathscr{H}}$ with $\operatorname{Tr}\rho = 1$; this describes an initial equilibrium with a heat bath at temperature β^{-1}. The magnetization of the nth site is given by

$$\langle\sigma_n^z(t)\rangle_M = \operatorname{Tr}(\rho\, e^{i\mathscr{H}_0 t}\sigma_n^z\, e^{-i\mathscr{H}_0 t}) \tag{2.5}$$

and thermalization is said to occur if

$$\lim_{t\to\infty}\lim_{M\to\infty}\langle\sigma_n^z(t)\rangle_M = 0 \tag{2.6}$$

since the equilibrium spontaneous magnetisation for $\mathscr{H}_0$ is known to be zero [2]. In (2.6) it is necessary to take the thermodynamic limit first in order to eliminate Poincaré-cyclic behavior. Analogous definitions hold for the correlation functions.

3. The isotropic case

The case when $\gamma = 0$ in (2.2) is non-trivial, since, although the total magnetisation is a constant of the motion, we have

$$[\mathscr{H}_0, \sigma_n^z] \neq 0. \tag{3.1}$$

The Hamiltonian $\mathscr{H}_0$ can be brought to diagonal form by the following well-known steps [1]:

1. Introduction of Fermi operators $c_m^\dagger$ by the Jordan–Wigner transformation

$$c_m^\dagger = (\sigma_m^x + i\sigma_m^y)\prod_{n=1}^{m-1}\exp[i\pi(\sigma_n^z + 1)/2]. \tag{3.2}$$

2. Spatial Fourier transformation, motivated by the symmetry introduced by the cyclic boundary conditions. We define

$$a_q^\dagger = M^{-1/2} \sum_1^M e^{iqm} c_m^\dagger. \tag{3.3}$$

The nature of the transformation (3.2), coupled with the cyclic boundary conditions, give the wavenumbers q a rather bizarre dependence on the parity of M and the parity of the fermion states on which $\mathscr{H}_0$ operates. The reader is assured that this recondite point has no effect in the thermodynamic limit; we shall therefore be rather cavalier about boundary conditions, assuming the q to be the M roots of

$$e^{iqM} = 1 \tag{3.4}$$

distinct modulo 2π, for all M.

The Hamiltonian is then given by

$$\mathscr{H}_0 = E_0 + \sum_q \cos q a_q^\dagger a_q. \tag{3.5}$$

From (3.2) and (3.3) we have

$$1 + \sigma_n^z = \frac{2}{M} \sum_{qq'} e^{in(q'-q)} a_q^\dagger a_{q'}. \tag{3.6}$$

Using (3.6), we obtain

$$e^{i\mathscr{H}_0 t}(1 + \sigma_n^z)\, e^{-i\mathscr{H}_0 t} = \frac{2}{M} \sum_{q,q'} e^{in(q'-q)}\, e^{it(\cos q - \cos q')} a_q^\dagger a_{q'}. \tag{3.7}$$

Since $\mathscr{H}$ becomes a quadratic form in the Fermi operators, it may be written in diagonal form as

$$\mathscr{H} = E_1 + \sum_j \lambda_j \alpha_j^\dagger \alpha_j \tag{3.8}$$

where the α_j are Fermi operators related to the a_q by a transformation

$$\alpha_j = \sum_q U_{jq} a_q \tag{3.9}$$

with a suitable unitary matrix U.

Evaluation of the trace in (2.5) is now straightforward:

$$\langle 1 + \sigma_n^z(t) \rangle_M = \frac{2}{M} \sum_{q,q'} e^{i[n(q'-q) + (\cos q - \cos q')t]} \sum_j U_{jq}^* U_{jq'} \langle \alpha_j^\dagger \alpha_j \rangle \tag{3.10}$$

where $\langle \alpha_j^\dagger \alpha_j \rangle$ is the Fermi occupation number given by

$$\langle \alpha_j^\dagger \alpha_j \rangle = (1 + e^{\beta \lambda_j})^{-1}. \tag{3.11}$$

The coefficients U_{jq} are determined by the eigenvalue problem

$$(\lambda - \cos q) U_{jq} = \frac{2h}{M} \sum_{q'} U_{jq'}\, e^{i(q-q')m}. \tag{3.12}$$

There are two possibilities:

(i) $\lambda_j = \cos q_0$ for some q_0 which solves (3.4). Then

$$\sum_q U_{jq}\, e^{-iqm} = 0 \tag{3.13}$$

and

$$U_{jq} = \frac{1}{\sqrt{2}}(\delta_{q,q_0}\, e^{iqm} - \delta_{q,-q_0}\, e^{-iqm}). \tag{3.14}$$

There are approximately $M/2$ distinct solutions of this class.

(ii) $\lambda_j \neq \cos q$: in this case

$$U_{jq} = e^{iqm}/N(\lambda_j)(\lambda_j - \cos q) \tag{3.15}$$

where the λ_j are zeros of the function

$$F_M(\lambda) = 1 - \frac{2h}{M}\sum_q (\lambda - \cos q)^{-1} \tag{3.16}$$

and the normalization constant $N(\lambda_j)$ is given by

$$|N(\lambda_j)|^2 = \sum_q (\lambda_j - \cos q)^{-2} = \frac{M}{2h}\frac{\partial F_M}{\partial \lambda}\bigg|_{\lambda=\lambda_j} \tag{3.17}$$

Graphical analysis of the function $F_M(\lambda)$ of (3.16) shows that there are approximately $M/2$ zeros. Thus we have all the solutions of (3.12).

Inserting the two different expressions for U_{jq} in (3.10) we obtain a sum of two parts, one from $\lambda_j \in (i)$ and one from $\lambda_j \in (ii)$. The latter can conveniently be rewritten as a contour integral, because by the residue theorem we have

$$\begin{aligned}\frac{M}{2h}\sum_{\lambda\in(ii)} \frac{1}{N^2(\lambda)(1+e^{\beta\lambda})(\lambda-\cos q)(\lambda-\cos q')} = \\ \frac{1}{2\pi i}\int_C \frac{d\lambda}{(1+e^{\beta\lambda})(\lambda-\cos q)(\lambda-\cos q')F_M(\lambda)} + \frac{M}{4h}\frac{(\delta_{qq'}+\delta_{-qq'})}{(1+e^{\beta\cos q})}\end{aligned} \tag{3.18}$$

where C is a contour enclosing the zeros of $F_M(\lambda)$ but none of the poles of $(1+e^{\beta\lambda})^{-1}$. In this way we get:

$$\langle 1 + \sigma_n^z(t)\rangle = I_M + S_M(t) \tag{3.19}$$

with

$$I_M = \frac{2}{M}\sum_q \frac{1}{1+e^{\beta\cos q}} \tag{3.20}$$

$$S_M(t) = \frac{2h}{\pi i}\oint_C \frac{G_M(\lambda, n-m, t)G_M(\lambda, m-n, -t)}{F_M(\lambda)(1+e^{\beta\lambda})}\, d\lambda \tag{3.21}$$

and

$$G_M(\lambda, n, t) = \frac{1}{M}\sum_q \frac{e^{-inq+it\cos q}}{(\lambda - \cos q)}. \tag{3.22}$$

The limit of these expressions as $M \to \infty$ can now easily be obtained:

$$I_\infty = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{dq}{1 + e^{\beta \cos q}} = 1 \tag{3.23}$$

$$G_\infty(\lambda, n, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-inq + it\cos q}}{(\lambda - \cos q)} dq \tag{3.24}$$

$$F_\infty(\lambda) = (1 - 2h(\lambda^2 - 1)^{-1/2})^{-1} \quad \text{with} \quad (\lambda^2 - 1)^{1/2} > 0 \quad \text{for} \quad \lambda > 0. \tag{3.25}$$

So

$$S_\infty(t) = \frac{2h}{\pi i}\int_C \frac{G_\infty(\lambda, n - m, t) G_\infty(\lambda, m - n, -t)}{F_\infty(\lambda)(1 + e^{\beta\lambda})} d\lambda. \tag{3.26}$$

We have used the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dq}{\lambda - \cos q} = (\lambda^2 - 1)^{-1/2}. \tag{3.27}$$

From these expressions it can be verified that

$$G_\infty(\lambda, n, t) = 0(t^{-1/2})$$

and thus that

$$\langle \sigma_n^z(t) \rangle_\infty = 0(t^{-1}) \qquad \text{as } t \to \infty.$$

4. Correlation functions

In this section we investigate the approach to equilibrium of the correlation functions for the isotropic Hamiltonian of (2.1) and (2.2). From (3.2) it is evident that any correlation function can be written as the trace of a product of the spins operators A_n and iB_n, collectively denoted Γ_j, where

$$A_n = c_n^\dagger + c_n, \qquad B_n = c_n^\dagger - c_n. \tag{4.1}$$

For instance the two-spin correlation functions are

$$\begin{aligned} \rho_{zz}(n, n + r) &= \langle A_n B_n A_{n+r} B_{n+r} \rangle \\ \rho_{xx}(n, n + r) &= \langle A_n B_n \cdots A_{n+r} B_{n+r} \rangle \\ \rho_{yy}(n, n + r) &= \langle B_n A_{n+1} \cdots A_{n+r} \rangle \end{aligned} \tag{4.2}$$

where for any operator K we define $\langle K \rangle$ by

$$\langle K \rangle = Tr(e^{-\beta\mathscr{H}} e^{i\mathscr{H}0t} K e^{-i\mathscr{H}0t})/Tr\, e^{-\beta\mathscr{H}}. \tag{4.3}$$

Since $\mathscr{H}0$ is a quadratic form in the fermions we may apply the Wick theorem to the evaluation of the correlation functions, which are then expressed in terms of the time-dependent contractions $\langle \Gamma_i \Gamma_j \rangle$ of the spinors Γ_i themselves. Using the methods of section 3, the $\langle \Gamma_i \Gamma_j \rangle$ may be evaluated directly, giving

$$\langle A_n A_{n+r} \rangle = 0(t^{-1}) \tag{4.4}$$

$$\langle A_n B_n \rangle = -\langle \sigma_n(t) \rangle \tag{4.5}$$

$$\langle A_n B_{n+r} \rangle = \frac{1}{\pi}\int_0^{\pi} dq \cos(qr) \tanh(\tfrac{1}{2}\beta \cos q) + f(n, n + r, t) \tag{4.6}$$

where the first term of (4.6) is the equilibrium value of $\langle A_n B_{n+r}\rangle$ for the Hamiltonian $\mathscr{H}_0$. The time-dependent part is given by

$$f(n, n+r, t) = \frac{2h}{\pi i}\int_C \frac{d\lambda}{F(\lambda)(1+e\beta\lambda)}(1 - 2h(\lambda^2 - 1)^{-1/2})^{-1}$$
$$\times\left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dq\, dq' \frac{\exp\{i[q(n+r) - q'n + (\cos q - \cos q')t]\}}{(\lambda - \cos q)(\lambda - \cos q')} \tag{4.7}$$

where the contour C does not surround the poles of $(1 + e^{\beta\lambda})^{-1}$. Using the analysis of Section 3, $f \sim t^{-1}$. Thus all correlation functions tend to their equilibrium values.

5. Anisotropic case

In Section 3 we derived the magnetization for $\gamma = 0$ in considerable detail. The case for $\gamma \neq 0$ is different in the sense that a Bogoliubov–Valatin transformation of the a_q is required to diagonalize $\mathscr{H}_0$. We rewrite (2.1) and (2.2) using the transformation of Section 3 as

$$\mathscr{H} = \sum_k \Lambda_k \alpha_k^\dagger \alpha_k + 2hM^{-1}\sum_{k,k'} b_k^\dagger b_{k'} \tag{5.1}$$

where we have taken $n = M$ without loss of generality because of the translational symmetry. In (5.1) we have

$$\Lambda_k^2 = 1 - (1 - \gamma^2)\sin^2 k \tag{5.2}$$

and the α_k are Fermi operators given by

$$b_k^\dagger = \alpha_k^\dagger \cos\phi_k - \alpha_{-k}\sin\phi_k. \tag{5.3}$$

The b_k are given in terms of the a_k of Section 3 by

$$b_k = e^{-\pi i/4} a_k. \tag{5.4}$$

The phase factor has been introduced to make the coefficients in (5.3) real. The transformation angle ϕ_k is given modulo π by

$$\cos 2\phi_k = \Lambda_k^{-1}\cos k$$
$$\sin 2\phi_k = -\Lambda_k^{-1}\gamma \sin k. \tag{5.5}$$

Notice that (5.3) is canonical, since ϕ_k is odd. Equation (5.1) can now be written

$$\mathscr{H} = \sum_k \Lambda_k \alpha_k^\dagger \alpha_k + \frac{2h}{M}\sum_{k,k'}\cos(\phi_k + \phi_{k'})\alpha_k^\dagger \alpha_{k'}$$
$$+ \frac{h}{M}\sum_{k,k'}\sin(\phi_{k'} - \phi_k)(\alpha_k^\dagger \alpha_{k'}^\dagger + \alpha_{k'}\alpha_k) \tag{5.6}$$

or

$$\mathscr{H} = \sum_{k,k'} A_{kk'}\alpha_k^+ \alpha_{k'} + \tfrac{1}{2}\sum_{k,k'}(\alpha_k^\dagger \alpha_{k'}^\dagger + \alpha_{k'}\alpha_k)B_{kk'}. \tag{5.7}$$

The matrices A and B may be written

$$A = E + 2hM^{-1}(|c\rangle\langle c| - |s\rangle\langle s|)$$
$$B = 2hM^{-1}(|c\rangle\langle s| - |s\rangle\langle c|) \tag{5.8}$$

with

$$E_{kk'} = \delta_{kk'}\Lambda_k. \tag{5.9}$$

The vectors $|c\rangle$ and $|s\rangle$ which occur in the projectors are defined by

$$\begin{aligned} \langle k|c\rangle &= \cos\phi_k \\ \langle k|s\rangle &= \sin\phi_k. \end{aligned} \tag{5.10}$$

The relationship of (5.7) to the analysis of Section 3 is quite clear; in the latter, the vectors $|s\rangle$ vanish, and $|c\rangle\langle c|$ becomes the unit operator. It is useful to define

$$\begin{aligned} |e\rangle &= |c\rangle - |s\rangle \\ |f\rangle &= |c\rangle + |s\rangle \end{aligned} \tag{5.11}$$

and the auxiliary matrices

$$\begin{aligned} X &= (A - B)(A + B) = E^2 + 2hM^{-1}(|f\rangle\langle e|E + E|e\rangle\langle f|) + 4h^2M^{-1}|e\rangle\langle e|) \\ Y &= (A + B)(A - B) = E^2 + 2hM^{-1}(|e\rangle\langle f|E + E|f\rangle\langle e|) + 4h^2M^{-1}|f\rangle\langle f|). \end{aligned} \tag{5.12}$$

We can now write $\mathscr{H}$ in diagonal form

$$\mathscr{H} = \sum_j \lambda_j \eta_j^\dagger \eta_j + \text{constant} \tag{5.13}$$

using the method described in [1] by performing a canonical transformation

$$\eta_j = \sum_k U_{jk}\alpha_k^\dagger + V_{jk}\alpha_k. \tag{5.14}$$

The U_{jk} and V_{jk} have to be chosen so that the quantities

$$\begin{aligned} x_k(\lambda_j) &= V_{jk} + U_{jk} \\ y_k(\lambda_j) &= V_{jk} - U_{jk} \end{aligned} \tag{5.15a}$$

are the normalized solutions of the eigenvalue equations

$$\begin{aligned} Xx(\lambda_j) &= \lambda_j^2 x(\lambda_j) \\ Yy(\lambda_j) &= \lambda_j^2 y(\lambda_j). \end{aligned} \tag{5.15b}$$

It is convenient to introduce some auxiliary functions:

$$\begin{aligned} \theta(\lambda_j^2) &= \frac{2h}{M}\sum_k f_k^2(\lambda_j^2 - \Lambda_k^2)^{-1} = \frac{2h}{M}\sum_k e_k^2(\lambda_j^2 - \Lambda_k^2)^{-1} \\ &= \frac{2h}{M}\sum_k (\lambda_j^2 - \Lambda_k^2)^{-1} \end{aligned} \tag{5.16a}$$

$$g(\lambda_j^2) = \frac{2h}{M}\sum_k f_k e_k \Lambda_k(\lambda_j^2 - \Lambda_k^2)^{-1} \tag{5.16b}$$

$$\sigma(\lambda_j^2) = \frac{2h}{M}\sum_k e_k^2\Lambda_k^2(\lambda_j^2 - \Lambda_k^2)^{-1} = \frac{2h}{M}\sum_k \Lambda_k^2(\lambda_j^2 - \Lambda_k^2)^{-1} \tag{5.16c}$$

where we made use of the relation

$$\phi_k = -\phi_{-k}.$$

The function $\theta(\lambda_j^2)$ and $\sigma(\lambda_j^2)$ are not independent, and one finds from (5.16) the relation

$$2h = -\sigma(\lambda_j^2) + \lambda_j^2\theta(\lambda_j^2). \tag{5.17}$$

We again find two cases:

(i) $\lambda_j^2 = \gamma^2 \sin^2 q + \cos^2 q$.

This expression has a four-fold degeneracy in q. We have

$$x_k(\lambda_j) = x_k^1\delta_{kq} + x_k^3\delta_{k,-q} + x_k^4\delta_{k}, + x_k^2\delta_{k,q-\pi} \tag{5.18}$$

which gives rise to 2 equations with 4 unknowns, i.e. two solutions:

$$\begin{gathered} x_q^1 - x_q^2 + x_q^3 - x_q^4 = 0 \\ f_q x_q^1 + f_q x_q^2 + e_q x_q^3 + e_q x_q^4 = 0 \end{gathered} \tag{5.19a}$$

$$\begin{gathered} y_q^1 - y_q^2 + y_q^3 - y_q^4 = 0 \\ f_q y_q^1 + f_q y_q^2 - e_q y_q^3 - e_q y_q^4 = 0. \end{gathered} \tag{5.19b}$$

The two solutions to these equations are easily obtained as

$$(x_q^1, x_q^2, x_q^3, x_q^4) \propto \begin{cases} \frac{1}{2}(e_q, e_q - f_q, -f_q) \\ \frac{1}{2}(1, -1, -1, 1) \end{cases} \tag{5.20a}$$

$$(y_q^1, y_q^2, y_q^3, y_q^4) \propto \begin{cases} \frac{1}{2}(e_q, e_q, f_q, f_q) \\ \frac{1}{2}(1, -1, -1, 1). \end{cases} \tag{5.20b}$$

and the U_{jq}, V_{jq} are determined by 5.15.

Case (ii) $\lambda_j^2 \neq \Lambda_k^2$ for all k.

By the method of the previous section we obtain

$$x_k(\lambda_j) = \left\{\frac{R(\lambda_j)f_k}{\lambda_j^2 - \Lambda_k^2} + \frac{S(\lambda_j)e_k\Lambda_k}{\lambda_j^2 - \Lambda_k^2}\right\}P_j^{-1} \tag{5.21}$$

where

$$S = \frac{2h}{M}\sum_k f_k x_k(\lambda_j) \tag{5.22a}$$

$$R = \frac{2h}{M}\sum_k (e_k f_k \Lambda_k + 2hf_k)x_k(\lambda_j) \tag{5.22b}$$

and by the use of (5.16) we obtain the linear equations

$$\begin{gathered} R = (2h\theta + g)R + (2hg + \sigma)S \\ S = \theta R + gS \end{gathered} \tag{5.23}$$

The condition for non-trivial solution, using relation (5.17) reads

$$(g-1)^2 - \lambda^2\theta^2 = 0. \tag{5.24}$$

So we have

$$\begin{pmatrix} R \\ S \end{pmatrix} \propto \begin{pmatrix} \lambda_j \\ 1 \end{pmatrix} \tag{5.25}$$

where λ_j is defined to have the same sign as $(1 - g)\theta^{-1}$. Define

$$F(\lambda_j) = -g(\lambda_j^2) - \lambda_j\theta(\lambda_j^2) + 1. \tag{5.26}$$

Then we obtain

$$x_k(x_j) = (\lambda_j f_k + \Lambda_k e_k)(\lambda_j^2 - \Lambda_k^2)^{-1} P_j^{-1} \tag{5.27a}$$

$$y_k(\lambda_j) = (\lambda_j e_k + \Lambda_k f_k)(\lambda_j^2 - \Lambda_k^2)^{-1} P_j^{-1} \tag{5.27b}$$

and P_j^2 is given as

$$\frac{dF}{d\lambda} = \frac{2h}{M} P_j^2. \tag{5.28}$$

Finally, the coefficients U_{jk}, V_{jk} are given by

$$U_{j,k} = \frac{c_k}{\lambda_j - \Lambda_k} P_j^{-1} \tag{5.29a}$$

$$V_{j,k} = \frac{s_k}{\lambda_j + \Lambda_k} P_j^{-1}. \tag{5.29b}$$

In order to compute the magnetization of nth spin at time t, when the mth spin is initially magnetized we express $1 + \sigma_r^z$ in terms of the new Fermi operators $\eta_j^\dagger, \eta_j$.

Using the same considerations as in Section 3, by the transformation (5.3) we obtain

$$(1 + \sigma_r^z)(t) = 2 \sum_{kk'ji} \{A_{kk'ij}\eta_i^\dagger\eta_j + B_{kk'ij}\eta_i\eta_j^\dagger + C_{kk'ij}\eta_i^\dagger\eta_j^\dagger + D_{kk'ij}\eta_i\eta_j\}. \tag{5.30}$$

When computing the expectation value of $1 + \sigma_r^z$ the last two terms do not contribute and may be omitted. The coefficients $A_{kk'ij}$, $B_{kk'ij}$ are given by

$$\begin{aligned} A_{kk'ij} = {} & \varphi(kk'rt)U_{ik}V_{jk'} + \psi(k, k', rt)V_{ik}V_{jk'} \\ & + \rho(kk'rt)U_{ik}V_{jk'} + \rho^*(k', k, rt)V_{ik}V_{jk'} \end{aligned} \tag{5.31a}$$

and

$$\begin{aligned} B_{kk'ij} = {} & \varphi(kk'rt)V_{ik}V_{jk'} + \psi(k, k', r, t)U_{ik}U_{jk'} \\ & + \rho(k, k', r, t)V_{ik}U_{jk'} + \rho^*(k', k, r, t)V_{ik}V_{jk'}. \end{aligned} \tag{5.31b}$$

We have

$$\varphi(kk'rt) = \exp(i(k - k')r + [\Lambda(k) - \Lambda(k')]t)c_k c_{k'} \tag{5.31c}$$

$$\psi(kk'rt) = \exp(i(k - k')r + [\Lambda(k) - \Lambda(k')]t)s_k s_{k'} \tag{5.31d}$$

and

$$\rho(kk'rt) = \exp(i(k + k')r + [\Lambda(k) + \Lambda(k')]t)c_k s_{k'}. \tag{5.31e}$$

To obtain the final answer we proceed same as before and divide (5.30) into three terms. Let $S(t)$ be again the time dependent term in (5.30). Taking the thermodynamic limit and converting the sum into a contour integral, we have

$$
\begin{aligned}
S(t) = \frac{h}{\pi i}\oint_c \frac{d\lambda}{F(\lambda)} \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int dk\, dk'\Bigg[(1+e^{\beta\lambda})^{-1}\bigg(\frac{\varphi(kk'rt)c(k)c(k')}{[\lambda-\Lambda(k)][\lambda-\Lambda(k')]} \\
+ \frac{\psi(kk'rt)s(k)s(k')}{[\lambda+\Lambda(k)][\lambda+\Lambda(k')]} + \frac{\rho(kk'rt)c(k)c(k')}{[\lambda-\Lambda(k)][\lambda+\Lambda(k')]} + \frac{\rho^*(k'krt)s(k)c(k')}{[\lambda+\Lambda(k)][\lambda-\Lambda(k')]}\bigg) \\
+ (1+e^{-\beta\lambda})^{-1}\bigg(\frac{\rho(kk'rt)s(k)s(k')}{[\lambda-\Lambda(k)][\lambda-\Lambda(k')]} + \frac{\psi(kk'rt)c(k)c(k')}{[\lambda+\Lambda(k)][\lambda+\Lambda(k')]} \\
+ \frac{\rho(kk'rt)s(k)c(k')}{[\lambda-\Lambda(k)][\lambda+\Lambda(k')]} + \frac{\rho^*(k'krt)c(k)s(k')}{[\lambda+\Lambda(k)][\lambda-\Lambda(k')]}\bigg)\Bigg].
\end{aligned}
\tag{5.32}
$$

$F(\lambda)$ is given by (5.26), and in the thermodynamic limit by

$$
F(\lambda) = 1 - \frac{h}{\pi}\int_{-\pi}^{\pi}\frac{dx(\lambda+\cos x)}{\lambda^2-\Lambda^2(x)}. \tag{5.33}
$$

The case of a constant magnetic field B applied to the system can be written down as a simple generalization of (5.32). The Hamiltonian in question is

$$
H_0 = \sum_i \{(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y - B\sigma_i^z\}. \tag{5.34}
$$

The result for the magnetization is given by (5.32) with each $\cos k$ being replaced by $\cos k - B$ plus a time-independent term.

The leading term in the asymptotic series of $S(t)$ for $t \sim \infty$ comes from the neighborhood of $(k, k') = (0, 0)$, (π, π) so only the first term of (5.32) contributes to the leading term in the asymptotic series.

Explicitly we have

$$
\begin{aligned}
S(t) \sim \frac{h}{\pi i}\int_C \frac{d\lambda}{F(\lambda)}(1+e^{-\beta\lambda})^{-1}\bigg\{\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\,\frac{e^{i[kr+\Lambda(k)t]}c^2(k)}{\lambda-\Lambda(k)} \\
\times \frac{1}{2\pi}\int_{-\pi}^{\pi} dk'\,\frac{e^{-i[k'r+\Lambda(k')t]}c^2(k')}{\lambda-\Lambda(k')}
\end{aligned}
\tag{5.35}
$$

where C is the same contour as in Section 3, with $F(\lambda)$ given by (5.33).

In (5.35) we use the definitions

$$
\Lambda(k) = [\gamma^2\sin^2 k + (\cos k - B)^2]^{1/2} \tag{5.36a}
$$

$$
F(\lambda) = 1 - \frac{h}{\pi}\int_{-\pi}^{\pi}\frac{dx[\lambda+(\cos x - B)]}{\lambda^2-\Lambda^2(x)} \tag{5.36b}
$$

$$
c(k) = \left[\frac{\cos k - B + \Lambda(k)}{2\Lambda(k)}\right]^{1/2}. \tag{5.36c}
$$

Note that $S(t)$ is again $O(t^{-1})$ for t large and $B \neq 1$. Combining these results we have for large t

$$
\langle\sigma_r^z(t)\rangle \cong \frac{1}{2\pi}\int_0^{\pi} dk\,\frac{\tanh[\frac{1}{2}\beta\Lambda]}{\Lambda}(\cos k - B) + O(t^{-1}) \tag{5.37}
$$

or in the limit

$$\lim_{t \to \infty} \langle \sigma_r(t) \rangle = \frac{1}{2\pi} \int_0^{\pi} dk \frac{\tanh[\frac{1}{2}\beta\Lambda]}{\Lambda} (\cos k - B) \tag{6.38}$$

which is the equilibrium result [2].

It is interesting to note that $S(t)$ approaches 0, in such a way, that the divisions into regions that was found earlier [3] does not occur here. In other words, $S(t) \sim t^{-1}$ for large t, for *all* $B \neq 1$.

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5. Private Communication.

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